Merging sequential e-values via martingales

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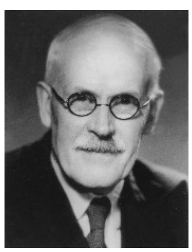

Users of these tests speak of the
5 per cent. point [p-value of 5%]
in much the same way as I should
speak of the $K = 10^{-1/2}$ point
[e-value of $10^{1/2}$], and of the 1
per cent. point [p-value of 1%]
as I should speak of the
$K = 10^{-1}$ point [e-value of 10].

Project "Hypothesis testing with e-values"

Working Paper #5

First posted 8 July 2020. Last revised October 19, 2020.

Project web site:
http://alrw.net/e

Abstract

We describe a general class of e-value merging functions via martingales, and prove its optimality in a few senses. We also describe a general class of ie-merging functions and pose the problem of finding ie-merging functions outside this class.

Contents

1 Introduction

This note continues discussion of an alternative, which we call e-values, to the standard statistical notion of p-values (e-values have been referred to as betting scores [6] and even, somewhat misleadingly, Bayes factors [7]). We concentrate on the problem of merging e-values; for a detailed wider discussion, see [1].

We will use the definitions given in our earlier paper [11] (however, we will reproduce some of those definitions). An ie-merging function is a function $F : [0,\infty)^K \to [0,\infty)$, for some $K \in \{2,3,\dots\}$ (fixed throughout this note), that maps K independent e-values $e_1,\dots,e_K$ to an e-value $F(e_1,\dots,e_K)$. This note introduces (in Section 4) a new class of ie-merging functions but does not contain any non-trivial mathematical results about this class.

It is interesting that the situations with merging p-values and e-values appear to be opposite. Merging independent p-values is in some sense trivial: for any measurable increasing function $F : [0,1]^K \to \mathbb{R}$ (intuitively, a test statistic), the function

$$G(p_1,\dots,p_K) := U(\{(q_1,\dots,q_K) \in [0,1]^K \mid F(q_1,\dots,q_K) \ge F(p_1,\dots,p_K)\}),$$

where U is the uniform probability measure on $[0,1]^K$, is an ip-merging function, and any ip-merging function can be obtained in this way. On the other hand, merging arbitrarily dependent p-values is difficult, in the sense that the structure of the class of all p-merging functions is very complicated (see, e.g., [10], including a review of previous results). In the case of e-values, merging arbitrarily dependent e-values is trivial, at least in the case of symmetric merging functions: according to [11, Proposition 3.1], arithmetic mean essentially dominates any symmetric e-merging function (and [11, Theorem 3.2] gives a full description of the class of all symmetric e-merging functions). Merging independent e-values is difficult and is the topic of this note.

We start in Section 2 from a subclass of the class of ie-merging functions, namely those that work for all sequential e-values. The definition is based on the idea of a martingale, and the game-theoretic version as defined in [8] is most convenient here. We discuss several interesting special cases.

In Section 4 we really need the independence of e-values. The notion of a martingale was introduced by Jean Ville [9] as extension (and correction) of von Mises's [5] notion of a gambling system. Kolmogorov [2] came up with another extension of von Mises's notion (later but independently a similar extension was proposed by Loveland [4, 3]). In Section 4 we combine Ville's and Kolmogorov's extensions to obtain our proposed class of ie-merging functions.

2 Merging sequential e-values

A function $F : [0,\infty)^K \to [0,\infty)$ is an *se-merging function* if, for any sequential e-variables $E_1,\dots,E_K$ on the same probability space, $F(E_1,\dots,E_K)$ is an e-variable on that probability space. Remember that e-variables $E_1,\dots,E_K$ are *sequential* if $\mathbb{E}[E_{k+1} \mid E_1,\dots,E_k] \le 1$ a.s., $k = 0,\dots,K-1$. If A is a measurable space, we let $A^{<K}$ stand for the measurable space $\cup_{k=0}^{K-1} A^k$, where $A^0 = \{\Box\}$, $\Box$ denoting the empty sequence.

A *gambling system* is a measurable function $s : [0,\infty)^{<K} \to [0,1]$. The *test martingale* associated with the gambling system s and initial capital $c \in [0,1]$ is the sequence of measurable functions $S_k : [0,\infty)^K \to [0,\infty)$, $k = 0,\dots,K$, which is defined recursively by $S_0 := c$ and

$$S_{k+1}(e_1,\dots,e_K) := S_k(e_1,\dots,e_K) \times \bigl(s(e_1,\dots,e_k)e_{k+1} + 1 - s(e_1,\dots,e_k)\bigr), \quad k = 0,\dots,K-1. \tag{1}$$

(This is a martingale in the generalized sense of [8].) The intuition is that we observe $e_1,\dots,e_K$ sequentially, start with capital at most 1, and at the end of step k invest a fraction $s(e_1,\dots,e_k)$ of our current capital in e_{k+1}, leaving the remaining capital aside. (We will also say that we *gamble* the fraction s of our capital and refer to s as our *bet*.) Then $S_k(e_1,\dots,e_K)$, which depends on $e_1,\dots,e_K$ only via $e_1,\dots,e_k$, is our resulting capital at time k.

Lemma 1. *A convex combination of test martingales is a test martingale.*

Proof. The statement of the lemma follows from the following equivalent (and often useful) definition: a test martingale is a sequence of nonnegative functions $S_k : [0,\infty)^K \to [0,\infty)$, $k = 0,\dots,K$, such that $S_0 \le 1$ and, for some measurable function $s : [0,\infty)^{<K} \to [0,\infty)$, we have

$$S_{k+1}(e_1,\dots,e_K) = S_k(e_1,\dots,e_K) + s(e_1,\dots,e_k)(e_{k+1} - 1) \tag{2}$$

for all $k = 0,\dots,K-1$ and all $e_1,\dots,e_K \in [0,\infty)$. ∎

A *martingale merging function* is a function $F : [0,\infty)^K \to [0,\infty)$ that can be represented in the form $F = S_K$ for some test martingale S_k, $k = 0,\dots,K$. The following two lemmas show that this is just a different definition of an se-merging function.

Lemma 2. *Any martingale merging function is an se-merging function.*

Proof. In our proofs, we will use the notation $\mathbb{P}$ for the probability measure in a probability space $(\Omega,\mathcal{F},\mathbb{P})$ that is clear from the context, and the notation $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$. A *filtration* in $(\Omega,\mathcal{F},\mathbb{P})$ is an increasing sequence $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_K$ of sub-σ-algebras of $\mathcal{F}$; we will set $\mathcal{F}_0 := \{\emptyset,\Omega\}$. We say that $(X_k,\mathcal{F}_k)$, $k = 0,\dots,K$, is a *martingale* on $(\Omega,\mathcal{F},\mathbb{P})$ if each X_k is a random variables on $(\Omega,\mathcal{F},\mathbb{P})$ that is $\mathcal{F}_k$-measurable and integrable and satisfies $\mathbb{E}(X_{k+1} \mid \mathcal{F}_k) = X_k$ for all $k = 0,\dots,K-1$.

Let $E_1,\dots,E_K$ be sequential e-variables in some probability space. It suffices to consider the case where $\mathbb{E}[E_k] = 1$ for each $k = 1,\dots,K$. Then $(S_k(E_1,\dots,E_K),\mathcal{F}_k)$, $k = 0,\dots,K$, where S_k is defined by (1) and $\mathcal{F}_k$ is the σ-algebra generated by $E_1,\dots,E_k$, is a martingale. This immediately implies $\mathbb{E}[S_K(E_1,\dots,E_K)] \le 1$. ∎

Theorem 1. *Any se-merging function is dominated by a martingale merging function.*

Proof. Let F be an se-merging function; our goal is to construct a dominating martingale merging function. Let $\mathcal{E}$ be the class of e-variables, i.e., nonnegative random variables E on the underlying probability space satisfying $\mathbb{E}(E) \le 1$.

First we consider e-variables taking values in the set $2^{-n}\mathbb{N}$, where $\mathbb{N} := \{0, 1, \dots\}$; let $\mathcal{E}_n$ be the set of such e-variables. Extend F to shorter sequences of e-values by

$$\begin{aligned} F_{n,K}(e_1, \dots, e_K) &:= F(e_1, \dots, e_K), \\ F_{n,k}(e_1, \dots, e_k) &:= \sup_{E \in \mathcal{E}_n} \mathbb{E}[F_{n,k+1}(e_1, \dots, e_k, E)] \end{aligned} \tag{3}$$

for all $e_1, \dots, e_K \in 2^{-n}\mathbb{N}$ and $k = K-1, \dots, 0$. It is clear that $F_0 \le 1$. By the duality theorem of linear programming, for any $k \in \{0, \dots, K-1\}$ and any $e_1, \dots, e_K \in 2^{-n}\mathbb{N}$, there exists $s \in [0,1]$ such that

$$\forall e \in 2^{-n}\mathbb{N} : F_{n,k+1}(e_1, \dots, e_k, e) \le F_{n,k}(e_1, \dots, e_k)(se + 1 - s). \tag{4}$$

Let us check carefully the application of the duality theorem. Let $c_1, \dots, c_N$ be the first N elements of the set $2^{-n}\mathbb{N}$ (namely, $c_i := (i-1)2^{-n}$, $i = 0, \dots, N-1$); we are interested in the case $N \to \infty$. Restricting E in (3) to take values $c_1, \dots, c_N$ with any probabilities $p_1, \dots, p_N$, instead of $F_{n,k}(e_1, \dots, e_k)$ we will obtain the solution $F_{n,k,N}$ to the linear programming problem

$$c_1 p_1 + \dots + c_N p_N \le 1 \tag{5}$$

$$p_1 + \dots + p_N = 1 \tag{6}$$

$$f_1 p_1 + \dots + f_N p_N \to \max, \tag{7}$$

where p_i are nonnegative variables and $f_i := F_{n,k+1}(e_1, \dots, e_k, c_i)$, $i = 1, \dots, N$. It is clear that the sequence $F_{n,k,N}$ is increasing in N and tends to $F_{n,k}(e_1, \dots, e_k)$ as $N \to \infty$. The dual problem to (5)–(7) is $y_1 + y_2 \to \min$ subject to $y_1 \ge 0$ and $c_i y_1 + y_2 \ge f_i$ for all $i \in \{1, \dots, N\}$. Then we will have the analogue

$$\forall e \in 2^{-n}\{0, \dots, N-1\} : F_{n,k+1}(e_1, \dots, e_k, e) \le F_{n,k,N}(se + 1 - s) \tag{8}$$

of (4) when $y_1 + y_2 = F_{n,k,N}$ (which is the case for the optimal (y_1, y_2)) and $y_1 = sF_{n,k,N}$. It is clear from (8) that $s \le 1$, as $e = 0$ is allowed. Let s_N be an s satisfying (8). Then any limit point of the sequence s_N will satisfy (4).

We have proved the statement of the theorem for e-values in $2^{-n}\mathbb{N}$; now we drop this assumption. Let $(e_1, \dots, e_K) \in [0, \infty)^K$. For each n, let $e_{n,k}$ be the largest number in $2^{-n}\mathbb{N}$ that does not exceed e_k. Set

$$F_k(e_1, \dots, e_k) := \lim_{n \to \infty} F_{n,k}(e_{n,1}, \dots, e_{n,k}),$$

Then F_k is a test martingale, and the fraction s to gamble after observing $e_1, \dots, e_k$ can be chosen as the smallest $s \in [0,1]$ satisfying

$$\forall e \in [0,\infty) : F_{k+1}(e_1, \dots, e_k, e) \le F_k(e_1, \dots, e_k)(se + 1 - s). \tag{9}$$

The set of such s is obviously closed; let us check that it is non-empty. Let $s = s_n$ be a number in $[0,1]$ satisfying (4) with $e_{n,1}, \dots, e_{n,k}$ in place of $e_1, \dots, e_k$, respectively. Then any limit point of s_n will satisfy (9). □

Examples of martingale merging functions

The simplest non-trivial gambling system is $s := 1$; the corresponding test martingale with initial capital 1 is the product

$$S_k(e_1, \dots, e_K) = e_1 \dots e_k,$$

and the corresponding martingale merging function is the product

$$F(e_1, \dots, e_K) := e_1 \dots e_K.$$

This is the most standard se-merging function.

Another martingale merging function is the arithmetic mean

$$F(e_1, \dots, e_K) := \frac{e_1 + \dots + e_K}{K}.$$

This is in fact an e-merging function (the most important symmetric one, as explained in [11]). The corresponding test martingale is the mean

$$S_k(e_1, \dots, e_K) := \frac{e_1 + \dots + e_k + K - k}{K}.$$

(This is easiest to see using the equivalent definition (2).)

A more general class of martingale merging functions, introduced in [11], includes the U-statistics

$$U_n(e_1, \dots, e_K) := \frac{1}{\binom{K}{n}} \sum_{\{k_1, \dots, k_n\} \subseteq \{1, \dots, K\}} e_{k_1} \dots e_{k_n}, \quad n \in \{0, 1, \dots, K\}. \tag{10}$$

This is a martingale merging function because each addend in (10) is, and a convex combination of test martingales is a test martingale (Lemma 1).

Our final martingale merging function has an increasing sequence of numbers $1 \le K_1 < \dots < K_m < K$ as its parameter and is defined as

$$F(e_1, \dots, e_K) := \prod_{i=0}^{m} \frac{e_{K_i+1} + \dots + e_{K_{i+1}}}{K_{i+1} - K_i},$$

where K_0 is understood to be 0 and K_{m+1} is understood to be K. The corresponding test martingale is

$$S_k(e_1, \dots, e_K) := \frac{e_1 + \cdots + e_{K_1}}{K_1} \cdots \frac{e_{K_{i-1}+1} + \cdots + e_{K_i}}{K_i - K_{i-1}} \frac{e_{K_i+1} + \cdots + e_k + K_{i+1} - k}{K_{i+1} - K_i},$$

where i is the largest number such that $K_i \le k$.

3 Merging sequential e-values into a martingale

In this section, we say that an e-variable is *precise* if $\mathbb{E}[E] = 1$, and an se-merging function F is *precise* if $\mathbb{E}[F(\mathbf{E})] = 1$ for any vector $\mathbf{E}$ of precise and sequential e-values. This property is satisfied by all examples of merging functions in [11]. All supermartingales, martingales and stopping times are with respect to the filtration generated by $\mathbf{E} = (E_1, \dots, E_K)$.

In scientific discovery, experiments are often conducted sequentially in time, and a discovery may be reported at the time when enough evidence is gathered. Therefore, with a vector $\mathbf{E}$ of sequential e-values, it is desirable to require validity of a test at not only the fixed time K, but also a stopping time τ. Such an approach produces an *anytime-valid* test. Fortunately, anytime validity is automatically achieved by using a test supermartingale: since $(S_k(\mathbf{E}))_{k=1,\dots,K}$ is a supermartingale, $S_\tau(\mathbf{E})$ is an e-variable for any stopping time τ.

Conversely, if a sequence of functions $F_k : [0, \infty)^K \to [0, \infty)$, $k = 0, 1, \dots, K$, satisfies

(a) *anytime validity*: $F_\tau(\mathbf{E})$ is an e-variable for any vector $\mathbf{E}$ of sequential e-values and any stopping time τ;

(b) *precision*: F_k is precise for each $k = 1, \dots, K$,

then we can show that it is a test martingale.

Theorem 2. *For a sequence of functions $F = (F_k)_{k=1,\dots,K}$, the following are equivalent:*

(i) F is a test martingale;

(ii) $F(\mathbf{E})$ is a martingale for any vector $\mathbf{E}$ of precise and sequential e-values;

(iii) F is anytime valid and precise; i.e., it satisfies (a) and (b).

Proof. The implications (i)⇒(ii) and (ii)⇒(iii) are straightforward. Below we show (iii)⇒(i).

Take any precise and sequential e-values $E_1, \dots, E_K$, and let $\mathcal{F} = (\mathcal{F}_k)_{k=1,\dots,K}$ be the natural filtration of $\mathbf{E} = (E_1, \dots, E_K)$.

Let τ be any ($\mathcal{F}$-)stopping time. First, we claim that $\mathbb{E}[F_\tau(\mathbf{E})] = 1$ holds. To show this claim, for $j = 1, \dots, K-1$ define $\tau_j = j$ if $\tau > j$ and $\tau_j = j + 1$

if $\tau \le j$. Clearly, τ_j is a stopping time for each j. Moreover, the realization of $(\tau, \tau_1, \dots, \tau_{K-1})$ is always a permutation of $(1, \dots, K)$. Hence, using (b), we have

$$\mathbb{E}\left[F_\tau(\mathbf{E}) + \sum_{j=1}^{K-1} F_{\tau_j}(\mathbf{E})\right] = \mathbb{E}\left[\sum_{k=1}^{K} F_k(\mathbf{E})\right] = K.$$

Using (a), $\mathbb{E}[F_{\tau_j}(\mathbf{E})] \le 1$ for each j. This implies $\mathbb{E}[F_\tau(\mathbf{E})] \ge 1$. Therefore, $\mathbb{E}[F_\tau(\mathbf{E})] = 1$ for any stopping time τ.

If for some $k = 1, \dots, K-1$, the event $A := \{\mathbb{E}[F_{k+1}(\mathbf{E}) \mid \mathcal{F}_k] > F_k(\mathbf{E})\}$ has a positive probability, then the stopping times $\eta := k1_A + K1_{A^c}$ and $\eta' := (k+1)1_A + K1_{A^c}$ satisfy $\mathbb{E}[F_\eta(\mathbf{E})] < \mathbb{E}[F_{\eta'}(\mathbf{E})]$, violating the property that $\mathbb{E}[F_\tau(\mathbf{E})] = 1$ for any stopping time τ. Hence, $\mathbb{P}(A) = 0$. Similarly, $\mathbb{P}(\mathbb{E}[F_{k+1}(\mathbf{E}) \mid \mathcal{F}_k] < F_k(\mathbf{E})) = 0$. Therefore, $\mathbb{E}[F_{k+1}(\mathbf{E}) \mid \mathcal{F}_k] = F_k(\mathbf{E})$ almost surely, and $(F_k(\mathbf{E}))_{k=1,\dots,K}$ is an $\mathcal{F}$-martingale.

Note that F_K is an se-merging function. By Theorem 1, we have $F_K(\mathbf{E}) \le S_K(\mathbf{E})$ for some test martingale $(S_k)_{k=1,\dots,K}$. Since $\mathbb{E}[F_K(\mathbf{E})] = \mathbb{E}[S_K(\mathbf{E})]$, we have $F_K(\mathbf{E}) = S_K(\mathbf{E})$ almost surely. Using the fact that both $(F_k(\mathbf{E}))_{k=1,\dots,K}$ and $(S_k(\mathbf{E}))_{k=1,\dots,K}$ are martingales, we have, for $k = 1, \dots, K$, almost surely

$$F_k(\mathbf{E}) = \mathbb{E}[F_K(\mathbf{E}) | \mathcal{F}_k] = \mathbb{E}[S_K(\mathbf{E}) | \mathcal{F}_k] = S_k(\mathbf{E}).$$

Since $\mathbf{E}$ is arbitrary, we have $F_K = S_K$. □

Theorem 2 implies that, in order to get an anytime-valid and precise method for merging sequential e-values, the only tool one could rely on is the class of test martingales.

4 Merging independent e-values

We can generalize the notion of a test martingale by allowing processing the input e-values in any order, following Kolmogorov [2, Section 2]. Let $\mathfrak{P}(A)$ be the class of all probability measures on a measurable space A. Consider measurable functions

$$p : \big(\{1, \dots, K\} \times [0,1] \times [0,\infty)\big)^{<K} \to \mathfrak{P}(\{1, \dots, K\}) \tag{11}$$

$$s : \big(\{1, \dots, K\} \times [0,1] \times [0,\infty)\big)^{<K} \times \{1, \dots, K\} \to \mathfrak{P}([0,1]), \tag{12}$$

where the first function satisfies

$$p(\pi_1, \sigma_1, e_1, \dots, \pi_k, \sigma_k, e_k)(\{\pi_1, \dots, \pi_k\}) = 0 \tag{13}$$

for all $k \in \{0, \dots, K-1\}$, all $\pi_1, \dots, \pi_k$, all $\sigma_1, \dots, \sigma_k$, and all $e_1, \dots, e_k$.

We generalize (1) to

$$\begin{aligned} S_{k+1}(e_1, \dots, e_K, \pi, \sigma) &:= S_k(e_1, \dots, e_K, \pi, \sigma) \\ &\quad \times \big(s(\pi(1), \sigma_1, e_{\pi(1)}, \dots, \pi(k), \sigma_k, e_{\pi(k)}) e_{\pi(k+1)} \\ &\qquad + 1 - s(\pi(1), \sigma_1, e_{\pi(1)}, \dots, \pi(k), \sigma_k, e_{\pi(k)})\big), \end{aligned} \tag{14}$$

where $\pi : \{1, \dots, K\} \to \{1, \dots, K\}$ is a permutation of the set $\{1, \dots, K\}$ and $\sigma = (\sigma_1, \dots, \sigma_K) \in [0,1]^K$ is a sequence of bets; as before, $S_0 \in [0,1]$. We say that F is a *generalized martingale merging function* F if there exist functions (11)–(12) such that, for all $e_1, \dots, e_K$,

$$F(e_1, \dots, e_K) = \mathbb{E} S_K(e_1, \dots, e_K, \pi, \sigma), \tag{15}$$

where $\mathbb{E}$ refers to the probability measure on the pairs (π, σ) (of permutations π of $\{1, \dots, K\}$ and sequences $\sigma \in [0,1]^K$) such that

- $p(\pi(1), \sigma_1, e_1, \dots, \pi(k), \sigma_k, e_k)$ is a version of the conditional distribution of $\pi(k+1)$ given $\pi(1), \sigma_1, \dots, \pi(k), \sigma_k$,
- and $s(\pi(1), \sigma_1, e_1, \dots, \pi(k), \sigma_k, e_k, \pi(k+1))$ is a version of the conditional distribution of σ_{k+1} given $\pi(1), \sigma_1, \dots, \pi(k), \sigma_k, \pi(k+1)$,

for all $k = 0, \dots, K-1$.

The intuition behind the definition (15) is that the value $F(e_1, \dots, e_K)$ is computed by gambling on $e_1, \dots, e_K$ in any order. In general, the gambling strategy is randomized, but let us first discuss the deterministic case and assume that the functions p and s in (11) and (12) always take values that are degenerate probability measures (i.e., those concentrated on a single point). Then (11)–(12) correspond to the following generalized (as compared to Section 2) gambling system, which uncovers the e-values in the order $e_{\pi(1)}, \dots, e_{\pi(K)}$ and gambles on each right before uncovering it. At the end of step k, $k = 1, \dots, K$, by which time we have seen the e-values $e_{\pi(1)}, \dots, e_{\pi(k)}$, we choose the index $\pi(k+1)$ of the next e-value to uncover using the function p (condition (13) ensuring that the e-value $e_{\pi(k+1)}$ has not been uncovered as yet). Namely, $\pi(k+1)$ is the point at which $p(\pi(1), \sigma_1, e_{\pi(1)}, \dots, \pi(k), \sigma_k, e_{\pi(k)})$ is concentrated. Right before uncovering $e_{\pi(k+1)}$, we invest a fraction $\sigma_{k+1} := s(\pi(1), \sigma_1, e_{\pi(1)}, \dots, \pi(k), \sigma_k, e_{\pi(k)})$ of our current capital in it, which gives us the resulting capital (14).

If the functions p and s in (11) and (12) are allowed to take non-degenerate values, the generalized gambling system becomes randomized, and the resulting merging function is obtained by averaging the merging functions corresponding to the realized π and σ. (Notice that in general the probability measure over which averaging is performed depends on $e_1, \dots, e_K$; the dependence, however, is non-anticipative in a natural sense.)

The generalized martingale merging functions form a subclass of the ie-merging functions, as the following lemma shows.

Lemma 3. *Any generalized martingale merging function is an ie-merging function.*

Proof. Let $E_1, \dots, E_K$ be independent e-variables on some probability space and F be a generalized martingale merging function. Our goal is to show that $F(E_1, \dots, E_K)$ is an e-variable.

We can extend the probability space in such a way that it carries random variables $E_1, \dots, E_K$ (with the same joint distribution as before), a random permutation π of $\{1, \dots, K\}$, and a random sequence $\sigma \in [0,1]^K$ such that

- $p(\pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)})$ is a version of the conditional distribution of $\pi(k+1)$ given $\pi(1)$, σ_1, $E_{\pi(1)}, \ldots$, $\pi(k)$, σ_k, $E_{\pi(k)}$,
- $s(\pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)}, \pi(k+1))$ is a version of the conditional distribution of σ_{k+1} given $\pi(1)$, σ_1, $E_{\pi(1)}, \ldots$, $\pi(k)$, σ_k, $E_{\pi(k)}$, $\pi(k+1)$,
- $E_{\pi(k+1)}, \ldots, E_{\pi(K)}$ are jointly independent of $\pi(1)$, σ_1, $E_{\pi(1)}, \ldots$, $\pi(k)$, σ_k, $E_{\pi(k)}$, $\pi(k+1)$, σ_{k+1},

for all $k = 0, \ldots, K-1$.

The proof proceeds by induction in K. Suppose the statement of the lemma holds for generalized martingale merging functions of $K-1$ arguments. We can compute the expected value of (15) in two steps as

$$\mathbb{E}F(E_1, \ldots, E_K) = \mathbb{E}S_K(E_1, \ldots, E_K, \pi, \sigma) = \mathbb{E}_1(\mathbb{E}_2 S_K(E_1, \ldots, E_K, \pi, \sigma)), \tag{16}$$

where

- the first $\mathbb{E}$ refers to the random choice of $E_1, \ldots, E_K$,
- the second $\mathbb{E}$ refers to the random choice of $E_1, \ldots, E_K, \pi, \sigma$,
- $\mathbb{E}_1$ refers to the random choice of $\pi(1)$, σ_1, and $E_{\pi(1)}$,
- and $\mathbb{E}_2$ refers to the random choice of the rest of π, σ, and E_k.

By the inductive assumption, $\mathbb{E}_2 S_K(E_1, \ldots, E_K, \pi, \sigma) = 1$, which implies $\mathbb{E}F(E_1, \ldots, E_K) = 1$. □

The following is an example of a generalized martingale merging function that is not an se-merging function.

Example 1. It is easy to check that

$$F(e_1, e_2) := \frac{1}{2}\left(\frac{e_1}{1+e_1} + \frac{e_2}{1+e_2}\right)(1 + e_1 e_2) \tag{17}$$

is an ie-merging function [11, Remark 4.3]. To see that F is also a generalized martingale merging function, notice that the symmetric expression (17) can be represented as the arithmetic average of

$$\frac{e_1}{1+e_1}(1 + e_1 e_2) = e_1\left(\frac{1}{1+e_1} + \frac{e_1}{1+e_1} e_2\right)$$

and the analogous expression with e_1 and e_2 interchanged. The generalized gambling strategy producing (17) starts from uncovering e_1 or e_2 with equal probabilities and investing all the capital in the chosen e-variable. If e_1 is uncovered first, it then invests a fraction of $e_1/(1+e_1)$ of its current capital into e_2. And if e_2 is uncovered first, it invests a fraction of $e_2/(1+e_2)$ of its current capital into e_1.

Let us now check that F is not an se-merging function. By the symmetry of F, we can assume, without loss of generality, that we first observe the e-variable E_1 producing e_1 and then observe E_2 producing e_2. Had F been an se-merging function,

$$\sup_{E_2 \in \mathcal{E}} \mathbb{E} F(e_1, E_2)$$

would have been an e-merging function of e_1. However, using the convexity of the functions $x/(1+x)$, x, and $x^2/(1+x)$ of $x \in [0, \infty)$, we obtain

$$\sup_{E_2 \in \mathcal{E}} \mathbb{E} F(e_1, E_2) = \max\left(e_1, \frac{e_1 + 1}{2}\right)$$

(the maximum is attained at E_2 that takes two values, one of which is 0), which is the maximum of two e-merging functions but not an e-merging function itself.

5 Conclusion

It remains an open question whether any ie-merging function is dominated by a generalized martingale merging function. We conjecture that the answer to this question is negative. It would be interesting to find simple, and perhaps even practically important, ie-merging functions not dominated by a generalized martingale merging function.

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