A class of ie-merging functions

Vladimir Vovk and Ruodu Wang

Users of these tests speak of the 5 per cent. point [p-value of 5%] in much the same way as I should speak of the $K = 10^{-1/2}$ point [e-value of $10^{1/2}$], and of the 1 per cent. point [p-value of 1%] as I should speak of the $K = 10^{-1}$ point [e-value of 10].

Project “Hypothesis testing with e-values”

Working Paper #5

July 8, 2020

Project web site:
http://alrw.net/e
Abstract

We describe a general class of ie-merging functions and pose the problem of finding ie-merging functions outside this class.

Contents

1  Introduction 1
2  Merging sequential e-values 1
3  Merging independent e-values 5
4  Conclusion 7
References 8
1 Introduction

This note continues discussion of an alternative, which we call e-values, to the standard statistical notion of p-values (e-values have been referred to as betting scores [6] and even, somewhat misleadingly, Bayes factors [7]). We concentrate on the problem of merging e-values; for a detailed wider discussion, see [1].

We will use the definitions given in our earlier paper [10] (however, we will reproduce some of those definitions). An ie-merging function is a function $F : [0, \infty)^K \to [0, \infty)$, for some $K \in \{2, 3, \ldots\}$ (fixed throughout this note), that maps $K$ independent e-values $e_1, \ldots, e_K$ to an e-value $F(e_1, \ldots, e_K)$. This note introduces (in Section 3) a new class of ie-merging functions but does not contain any non-trivial mathematical results about this class.

It is interesting that the situations with merging p-values and e-values appear to be opposite. Merging independent p-values is in some sense trivial: for any measurable increasing function $F : [0, 1]^K \to \mathbb{R}$ (intuitively, a test statistic), the function $G(p_1, \ldots, p_K) := U\left(\{ (q_1, \ldots, q_K) \in [0,1]^K \mid F(q_1, \ldots, q_K) \geq F(p_1, \ldots, p_K) \}\right)$, where $U$ is the uniform probability measure on $[0,1]^K$, is an ip-merging function, and any ip-merging function can be obtained in this way. On the other hand, merging arbitrarily dependent p-values is difficult, in the sense that the structure of the class of all p-merging functions is very complicated (see, e.g., [11], including a review of previous results). In the case of e-values, merging arbitrarily dependent e-values is trivial, at least in the case of symmetric merging functions: according to [10, Proposition 3.1], arithmetic mean essentially dominates any symmetric e-merging function (and [10, Theorem 3.2] gives a full description of the class of all symmetric e-merging functions). Merging independent e-values is difficult and is the topic of this note.

We start in Section 2 from a subclass of the class of ie-merging functions, namely those that work for all sequential e-values. The definition is based on the idea of a martingale, and the game-theoretic version as defined in [8] is most convenient here. We discuss several interesting special cases.

In Section 3 we really need the independence of e-values. The notion of a martingale was introduced by Jean Ville [9] as extension (and correction) of von Mises’s [5] notion of a gambling system. Kolmogorov [2] came up with another extension of von Mises’s notion (later but independently a similar extension was proposed by Loveland [4, 3]). In Section 3 we combine Ville’s and Kolmogorov’s extensions to obtain our proposed class of ie-merging functions.

2 Merging sequential e-values

A function $F : [0, \infty)^K \to [0, \infty)$ is an se-merging function if, for any sequential e-variables $E_1, \ldots, E_K$ on the same probability space, $F(E_1, \ldots, E_K)$ is an e-variable on that probability space. Remember that e-variables $E_1, \ldots, E_K$ are sequential if $E[E_{k+1} \mid E_1, \ldots, E_k] \leq 1$ a.s., $k = 0, \ldots, K-1$. If $A$ is measurable
space, we let $A^{< K}$ stand for the measurable space $\bigcup_{k=0}^{K-1} A^k$, where $A^0 = \emptyset$, $\emptyset$ denoting the empty sequence.

A gambling system is a measurable function $s : [0, \infty)^{< K} \to [0, 1]$. The test martingale associated with the gambling system $s$ and initial capital $c \in [0, 1]$ is the sequence of measurable functions $S_k : [0, \infty)^K \to [0, \infty)$, $k = 0, \ldots, K$, which is defined recursively by $S_0 := c$ and

$$S_{k+1}(e_1, \ldots, e_K) := S_k(e_1, \ldots, e_K) \times (s(e_1, \ldots, e_k)e_{k+1} + 1 - s(e_1, \ldots, e_k)), \quad k = 0, \ldots, K - 1.$$  

(This is a martingale in the generalized sense of [8].) The intuition is that we observe $e_1, \ldots, e_K$ sequentially, start with capital at most 1, and at the end of step $k$ invest a fraction $s(e_1, \ldots, e_k)$ of our current capital in $e_{k+1}$, leaving the remaining capital aside. (We will also say that we gamble the fraction $s$ of our capital and refer to $s$ as our bet.) Then $S_k(e_1, \ldots, e_K)$, which depends on $e_1, \ldots, e_k$ only via $e_1, \ldots, e_k$, is our resulting capital at time $k$.

**Lemma 1.** A convex combination of test martingales is a test martingale.

**Proof.** The statement of the lemma follows from the following equivalent (and often useful) definition: a test martingale is a sequence of nonnegative functions $S_k : [0, \infty)^K \to [0, \infty)$, $k = 0, \ldots, K$, such that $S_0 \leq 1$ and, for some measurable function $s : [0, \infty)^{< K} \to [0, \infty)$, we have

$$S_{k+1}(e_1, \ldots, e_K) = S_k(e_1, \ldots, e_K) + s(e_1, \ldots, e_k)(e_{k+1} - 1)$$  

for all $k = 0, \ldots, K - 1$ and all $e_1, \ldots, e_K \in [0, \infty)$.

A martingale merging function is a function $F : [0, \infty)^K \to [0, \infty)$ that can be represented in the form $F = S_k$ for some test martingale $S_k$, $k = 0, \ldots, K$. The following two lemmas show that this is just a different definition of an se-merging function.

**Lemma 2.** Any martingale merging function is an se-merging function.

**Proof.** In our proofs, we will use the notation $\mathbb{P}$ for the probability measure in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is clear from the context, and the notation $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$. A filtration in $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing sequence $\mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_K$ of sub-$\sigma$-algebras of $\mathcal{F}$; we will set $\mathcal{F}_0 := \{\emptyset, \Omega\}$. We say that $(X_k, \mathcal{F}_k)$, $k = 0, \ldots, K$, is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ if each $X_k$ is a random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ that is $\mathcal{F}_k$-measurable and integrable and satisfies $\mathbb{E}(X_{k+1} \mid \mathcal{F}_k) = X_k$ for all $k = 0, \ldots, K - 1$.

Let $E_1, \ldots, E_K$ be sequential $\varepsilon$-variables in some probability space. Then $(S_k(E_1, \ldots, E_K), \mathcal{F}_k)$, $k = 0, \ldots, K$, where $S_k$ is defined by (1) and $\mathcal{F}_k$ is the $\sigma$-algebra generated by $E_1, \ldots, E_k$, is a martingale. This immediately implies $\mathbb{E}S_K(E_1, \ldots, E_K) \leq 1$.

**Lemma 3.** Any se-merging function is dominated by a martingale merging function.
Proof. Let $F$ be an $e$-merging function; our goal is to construct a dominating martingale merging function. Let $\mathcal{E}$ be the class of $e$-variables, i.e., nonnegative random variables $E$ on the underlying probability space satisfying $\mathbb{E}(E) \leq 1$.

First we consider $e$-variables taking values in the set $2^{-N}\mathbb{N}$, where $N := \{0, 1, \ldots\}$; let $\mathcal{E}_n$ be the set of such $e$-variables. Extend $F$ to shorter sequences of $e$-values by

$$F_{n,K}(e_1, \ldots, e_K) := F(e_1, \ldots, e_K),$$
$$F_{n,k}(e_1, \ldots, e_k) := \sup_{E \in \mathcal{E}_n} EF_{n,k+1}(e_1, \ldots, e_k, E)$$

for all $e_1, \ldots, e_K \in 2^{-N}\mathbb{N}$ and $k = K - 1, \ldots, 0$. It is clear that $F_0 \leq 1$. By the duality theorem of linear programming, for any $k \in \{0, \ldots, K - 1\}$ and any $e_1, \ldots, e_K \in 2^{-N}\mathbb{N}$, there exists $s \in [0, 1]$ such that

$$\forall e \in 2^{-N}\mathbb{N} : F_{n,k+1}(e_1, \ldots, e_k, e) \leq F_{n,k}(e_1, \ldots, e_k)(se + 1 - s).$$

Let us check carefully the application of the duality theorem. Let $c_1, \ldots, c_N$ be the first $N$ elements of the set $2^{-N}\mathbb{N}$ (namely, $c_i := (i-1)2^{-n}$, $i = 0, \ldots, N - 1$); we are interested in the case $N \to \infty$. Restricting $E$ in (3) to take values $e_1, \ldots, e_N$ with any probabilities $p_1, \ldots, p_N$, instead of $F_{n,k}(e_1, \ldots, e_k)$ we will obtain the solution $F_{n,k,N}$ to the linear programming problem

$$c_1 p_1 + \cdots + c_N p_N \leq 1$$
$$p_1 + \cdots + p_N = 1$$
$$f_1 p_1 + \cdots + f_N p_N \to \max,$$

where $p_i$ are nonnegative variables and $f_i := F_{n,k+1}(e_1, \ldots, e_k, c_i)$, $i = 1, \ldots, N$. It is clear that the sequence $F_{n,k,N}$ is increasing in $N$ and tends to $F_{n,k}(e_1, \ldots, e_k)$ as $N \to \infty$. The dual problem to (5)–(7) is

$$y_1 + y_2 \to \min$$
$$y_1 \geq 0$$
$$c_i y_1 + y_2 \geq f_i$$

for all $i \in \{1, \ldots, N\}$. Then we will have the analogue

$$\forall e \in 2^{-n}\{0, \ldots, N - 1\} : F_{n,k+1}(e_1, \ldots, e_k, e) \leq F_{n,k,N}(se + 1 - s)$$

of (4) when $y_1 + y_2 = F_{n,k,N}$ (which is the case for the optimal $(y_1, y_2)$) and $y_1 = seF_{n,k,N}$. It is clear from (8) that $s \leq 1$, as $e = 0$ is allowed. Let $s_N$ be an $s$ satisfying (8). Then any limit point of the sequence $s_N$ will satisfy (4).

We have proved the statement of the theorem for $e$-values in $2^{-N}\mathbb{N}$; now we drop this assumption. Let $(e_1, \ldots, e_K) \in [0, \infty)^K$. For each $n$, let $e_{n,k}$ be the largest number in $2^{-N}\mathbb{N}$ that does not exceed $e_k$. Set

$$F_k(e_1, \ldots, e_k) := \lim_{n \to \infty} F_{n,k}(e_{n,1}, \ldots, e_{n,k}),$$

Then $F_k$ is a test martingale, and the fraction $s$ to gamble after observing $e_1, \ldots, e_k$ can be chosen as the smallest $s \in [0, 1]$ satisfying

$$\forall e \in [0, \infty) : F_{k+1}(e_1, \ldots, e_k, e) \leq F_k(e_1, \ldots, e_k)(se + 1 - s).$$

The set of such $s$ is obviously closed; let us check that it is non-empty. Let $s = s_n$ be a number in $[0, 1]$ satisfying (4) with $e_{n,1}, \ldots, e_{n,k}$ in place of $e_1, \ldots, e_k$, respectively. Then any limit point of $s_n$ will satisfy (9).
Examples of martingale merging functions

The simplest non-trivial gambling system is \( s := 1 \); the corresponding test martingale with initial capital 1 is the product

\[
S_k(e_1, \ldots, e_K) = e_1 \ldots e_k,
\]

and the corresponding martingale merging function is the product

\[
F(e_1, \ldots, e_K) := e_1 \ldots e_K.
\]

This is the most standard se-merging function.

Another martingale merging function is the arithmetic mean

\[
F(e_1, \ldots, e_K) := \frac{e_1 + \cdots + e_K}{K}.
\]

This is in fact an e-merging function (the most important symmetric one, as explained in [10]). The corresponding test martingale is the mean

\[
S_k(e_1, \ldots, e_K) := \frac{e_1 + \cdots + e_k + K - k}{K}.
\]

(This is easiest to see using the equivalent definition (2).)

A more general class of martingale merging functions, introduced in [10], includes the U-statistics

\[
U_n(e_1, \ldots, e_K) := \frac{1}{\binom{K}{n}} \sum_{\{k_1, \ldots, k_n\} \subseteq \{1, \ldots, K\}} e_{k_1} \cdots e_{k_n}, \quad n \in \{0, 1, \ldots, K\}.
\] (10)

This is a martingale merging function because each addend in (10) is, and a convex combination of test martingales is a test martingale (Lemma 1).

Our final martingale merging function has an increasing sequence of numbers \( 1 \leq K_1 < \cdots < K_m < K \) as its parameter and is defined as

\[
F(e_1, \ldots, e_K) := \prod_{i=0}^{m} \frac{e_{K_i+1} + \cdots + e_{K_{i+1}}}{K_{i+1} - K_i},
\]

where \( K_0 \) is understood to be 0 and \( K_{m+1} \) is understood to be \( K \). The corresponding test martingale is

\[
S_k(e_1, \ldots, e_K) := \frac{e_1 + \cdots + e_{K_i}}{K_i} \cdot \frac{e_{K_{i-1}+1} + \cdots + e_{K_i}}{K_i - K_{i-1}} \cdot \frac{e_{K_{i+1}+1} + \cdots + e_{k + K_{i+1} - K_i}}{K_{i+1} - K_i},
\]

where \( i \) is the largest number such that \( K_i \leq k \).
3 Merging independent e-values

We can generalize the notion of a test martingale by allowing processing the input e-values in any order, following Kolmogorov [2, Section 2]. Let \( \mathcal{P}(A) \) be the class of all probability measures on a measurable space \( A \). Consider measurable functions

\[
p : (\{1, \ldots, K\} \times [0, 1] \times [0, \infty))^<K \to \mathcal{P}\{1, \ldots, K\}
\]

\[
s : (\{1, \ldots, K\} \times [0, 1] \times [0, \infty))^<K \times \{1, \ldots, K\} \to \mathcal{P}([0, 1]),
\]

where the first function satisfies

\[
p(\pi_1, \sigma_1, e_1, \ldots, \pi_k, \sigma_k, e_k)(\{\pi_1, \ldots, \pi_k\}) = 0
\]

for all \( k \in \{0, \ldots, K - 1\} \), all \( \pi_1, \ldots, \pi_k \), all \( \sigma_1, \ldots, \sigma_k \), and all \( e_1, \ldots, e_k \).

We generalize (1) to

\[
S_{k+1}(e_1, \ldots, e_K, \pi, \sigma) := S_k(e_1, \ldots, e_K, \pi, \sigma) \times (s(\pi(1), \sigma_1, e_{\pi(1)}, \ldots, \pi(k), \sigma_k, e_{\pi(k)})) e_{\pi(k+1)} \times 1 - s(\pi(1), \sigma_1, e_{\pi(1)}, \ldots, \pi(k), \sigma_k, e_{\pi(k)}),
\]

where \( \pi : \{1, \ldots, K\} \to \{1, \ldots, K\} \) is a permutation of the set \( \{1, \ldots, K\} \) and \( \sigma = (\sigma_1, \ldots, \sigma_K) \in [0, 1]^K \) is a sequence of bets; as before, \( S_0 \in [0, 1] \). We say that \( F \) is a generalized martingale merging function \( F \) if there exist functions (11)–(12) such that, for all \( e_1, \ldots, e_K \),

\[
F(e_1, \ldots, e_K) = ES_K(e_1, \ldots, e_K, \pi, \sigma),
\]

where \( E \) refers to the probability measure on the pairs \( (\pi, \sigma) \) (of permutations \( \pi \) of \( \{1, \ldots, K\} \) and sequences \( \sigma \in [0, 1]^K \)) such that

- \( p(\pi(1), \sigma_1, e_1, \ldots, \pi(k), \sigma_k, e_k) \) is a version of the conditional distribution of \( \pi(k+1) \) given \( \pi(1), \sigma_1, \ldots, \pi(k), \sigma_k \),
- and \( s(\pi(1), \sigma_1, e_1, \ldots, \pi(k), \sigma_k, e_k, \pi(k+1)) \) is a version of the conditional distribution of \( \sigma_{k+1} \) given \( \pi(1), \sigma_1, \ldots, \pi(k), \sigma_k, \pi(k+1) \),

for all \( k = 0, \ldots, K - 1 \).

The intuition behind the definition (15) is that the value \( F(e_1, \ldots, e_K) \) is computed by gambling on \( e_1, \ldots, e_K \) in any order. In general, the gambling strategy is randomized, but let us first consider the deterministic case and assume that the functions \( p \) and \( s \) in (11) and (12) always take values that are degenerate probability measures (i.e., those concentrated on a single point). Then (11)–(12) correspond to the following generalized (as compared to Section 2) gambling system, which uncovers the e-values in the order \( e_{\pi(1)}, \ldots, e_{\pi(K)} \) and gambles on each right before uncovering it. At the end of step \( k, k = 1, \ldots, K \), by which time we have seen the e-values \( e_{\pi(1)}, \ldots, e_{\pi(k)} \),
we choose the index \( \pi(k + 1) \) of the next e-value to uncover using the function \( p \) (condition (13) ensuring that the e-value \( e_{\pi(k+1)} \) has not been uncovered as yet). Namely, \( \pi(k+1) \) is the point at which \( p(\pi(1), \sigma_1, e_{\pi(1)}, \ldots, \pi(k), \sigma_k, e_{\pi(k)}) \) is concentrated. Right before uncovering \( e_{\pi(k+1)} \), we invest a fraction \( \sigma_{k+1} := s(\pi(1), \sigma_1, e_{\pi(1)}, \ldots, \pi(k), \sigma_k, e_{\pi(k)}) \) of our current capital in it, which gives us the resulting capital (14).

If the functions \( p \) and \( s \) in (11) and (12) are allowed to take non-degenerate values, the generalized gambling system becomes randomized, and the resulting merging function is obtained by averaging the merging functions corresponding to the realized \( \pi \) and \( \sigma \). (Notice that in general the probability measure over which averaging is performed depends on \( e_1, \ldots, e_K \); the dependence, however, is non-anticipative in a natural sense.)

The generalized martingale merging functions form a subclass of the ie-merging function, as the following lemma shows.

**Lemma 4.** Any generalized martingale merging function is an ie-merging function.

**Proof.** Let \( E_1, \ldots, E_K \) be independent e-variables on some probability space and \( F \) be a generalized martingale merging function. Our goal is to show that \( F(E_1, \ldots, E_K) \) is an e-variable.

We can extend the probability space in such a way that it carries random variables \( E_1, \ldots, E_K \) (with the same joint distribution as before), a random permutation \( \pi \) of \( \{1, \ldots, K\} \), and a random sequence \( \sigma \in [0, 1]^K \) such that

- \( p(\pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)}) \) is a version of the conditional distribution of \( \pi(k+1) \) given \( \pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)} \),
- \( s(\pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)}, \pi(k+1)) \) is a version of the conditional distribution of \( \sigma_{k+1} \) given \( \pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)}, \pi(k+1) \),
- \( E_{\pi(k+1)}, \ldots, E_{\pi(K)} \) are jointly independent of \( \pi(1), \sigma_1, E_{\pi(1)}, \ldots, \pi(k), \sigma_k, E_{\pi(k)}, \pi(k+1), \sigma_{k+1} \),

for all \( k = 0, \ldots, K - 1 \).

The proof proceeds by induction in \( K \). Suppose the statement of the lemma holds for generalized martingale merging functions of \( K-1 \) arguments. We can compute the expected value of (15) in two steps as

\[
\mathbb{E} F(E_1, \ldots, E_K) = \mathbb{E} S_K(E_1, \ldots, E_K, \pi, \sigma) = \mathbb{E}_1(\mathbb{E}_2 S_K(E_1, \ldots, E_K, \pi, \sigma)),
\]

where

- the first \( \mathbb{E} \) refers to the random choice of \( E_1, \ldots, E_K \),
- the second \( \mathbb{E} \) refers to the random choice of \( E_1, \ldots, E_K, \pi, \sigma \),
- \( \mathbb{E}_1 \) refers to the random choice of \( \pi(1), \sigma_1, \) and \( E_{\pi(1)} \).
• and $E_2$ refers to the random choice of the rest of $\pi$, $\sigma$, and $E_k$.

By the inductive assumption, $E_2S_K(E_1, \ldots, E_K, \pi, \sigma) = 1$, which implies $\mathbb{E}F(E_1, \ldots, E_K) = 1$. \hfill \Box

The following is an example of a generalized martingale merging function that is not an se-merging function.

**Example 1.** It is easy to check that

$$F(e_1, e_2) := \frac{1}{2} \left( \frac{e_1}{1 + e_1} + \frac{e_2}{1 + e_2} \right) (1 + e_1 e_2) \quad (17)$$

is an ie-merging function [10, Remark 4.3]. To see that $F$ is also a generalized martingale merging function, notice that the symmetric expression (17) can be represented as the arithmetic average of

$$\frac{e_1}{1 + e_1} (1 + e_1 e_2) = e_1 \left( \frac{1}{1 + e_1} + \frac{e_1}{1 + e_1} \right)$$

and the analogous expression with $e_1$ and $e_2$ interchanged. The generalized gambling strategy producing (17) starts from uncovering $e_1$ or $e_2$ with equal probabilities and investing all the capital in the chosen e-variable. If $e_1$ is uncovered first, it then invests a fraction of $e_1/(1 + e_1)$ of its current capital into $e_2$. And if $e_2$ is uncovered first, it invests a fraction of $e_2/(1 + e_2)$ of its current capital into $e_1$.

Let us now check that $F$ is not an se-merging function. By the symmetry of $F$, we can assume, without loss of generality, that we first observe the e-variable $E_1$ producing $e_1$ and then observe $E_2$ producing $e_2$. Had $F$ been an se-merging function,

$$\sup_{E_2 \in \mathcal{E}} \mathbb{E}F(e_1, E_2)$$

would have been an e-merging function of $e_1$. However, using the convexity of the functions $x/(1 + x)$, $x$, and $x^2/(1 + x)$ of $x \in [0, \infty)$, we obtain

$$\sup_{E_2 \in \mathcal{E}} \mathbb{E}F(e_1, E_2) = \max \left( e_1, \frac{e_1 + 1}{2} \right)$$

(the maximum is attained at $E_2$ that takes two values, one of which is 0), which is the maximum of two e-merging functions but not an e-merging function itself.

### 4 Conclusion

It remains an open question whether any ie-merging function is dominated by a generalized martingale merging function. We conjecture that the answer to this question is negative. It would be interesting to find simple, and perhaps even practically important, ie-merging functions not dominated by a generalized martingale merging function.
References


