Admissible ways of merging p-values under arbitrary dependence

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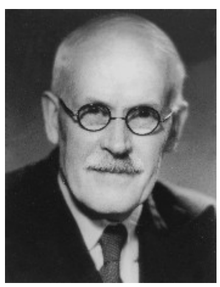

Users of these tests speak of the
5 per cent. point [p-value of 5%]
in much the same way as I should
speak of the $K = 10^{-1/2}$ point
[e-value of $10^{1/2}$], and of the 1
per cent. point [p-value of 1%]
as I should speak of the
$K = 10^{-1}$ point [e-value of 10].

Project "Hypothesis testing with e-values"

Working Paper #6

First posted July 28, 2020. Last revised March 29, 2021.

Project web site:
http://alrw.net/e

Abstract

Methods of merging several p-values into a single p-value are important in their own right and widely used in multiple hypothesis testing. This paper is the first to systematically study the admissibility (in Wald's sense) of p-merging functions and their domination structure, without any information on the dependence structure of the input p-values. As a technical tool we use the notion of e-values, which are alternatives to p-values recently promoted by several authors. We obtain several results on the representation of admissible p-merging functions via e-values and on (in)admissibility of existing p-merging functions. By introducing new admissible p-merging functions, we show that some classic merging methods can be strictly improved to enhance power without compromising validity under arbitrary dependence.

Contents

1 Introduction

A common task in multiple testing of a single hypothesis and testing multiple hypotheses is to combine several p-values into one p-value (without using the underlying data). If one assumes independence (or another specific dependence structure) among p-values testing a scientific hypothesis H_0, then the combined p-value is effectively testing a composition of H_0 and the independence assumption. A rejection obtained from such a test may be due to statistical evidence against either independence or the scientific hypothesis of interest (or both). As we typically only have one realization of a bunch of p-values, it is not possible to identify the source of rejection. Hence, such a method cannot be justified unless convincing evidence of independence is supplied; however, as argued by Efron [2010, pp. 50–51], neither independence nor positive regression dependence, which is often assumed in literature, is realistic in large-scale inference. Therefore, it is important to consider merging methods that are valid without available information on the dependence structure. In general, dropping the assumption of independence makes the problem of merging p-values more difficult: see, e.g., Vovk and Wang [2020c, Section 1].

Several valid merging methods are known for arbitrary dependence structure among p-values; these methods do not make any other assumptions about the input p-values (such as assumptions about their support; those p-values can be continuous or discrete), and their validity is exact (and not, e.g., asymptotic or approximate). Of course, such methods, which we will call *universally valid*, come at a cost of power. The most well-known one is arguably the Bonferroni correction, which uses the minimum of p-values times the number of tests. Several other methods include those of Rüger [1978] and Hommel [1983], based on order statistics of p-values, and those of Vovk and Wang [2020a], based on generalized means of p-values; see Section 3 for details of these merging methods. These methods include versions of the method of Simes [1986] and the harmonic mean of Wilson [2019] that are adjusted to be valid under arbitrary dependence.

Our study gives rise to new universally valid merging methods (in particular, free of any dependence assumptions) that are more powerful than the ones in the existing literature. Perhaps the main of these methods is what we call the grid harmonic method H_K^*, which improves on the method of Hommel [1983]. Our simulation studies demonstrate that the improvement is very substantial, which shows in applications that are important in practice, such as multiple hypothesis testing. See Sections 7 and 10.

The main objective of this paper is to study the domination structure among universally valid functions for merging p-values, henceforth *p-merging functions*. In particular, we do not discuss methods that are valid for specific classes of dependence structures; for the latter, see e.g., Sarkar [1998], Wilson [2019], and Liu and Xie [2020], as well as Chen et al. [2020] for a summary. A p-merging function is *admissible* if it is not strictly dominated by any other p-merging function. Ideally, ceteris paribus, only admissible p-merging functions should be used, as other methods can be strictly improved. It turns out that admissibility and domination structure among p-merging functions give rise to highly non-trivial mathematical challenges. We are mainly interested in homogeneous and symmetric p-merging functions, as most p-merging functions used in practice are of this kind.

Let us briefly summarize our main contributions. First, the merging function of Simes [1986] (valid under the assumption of independence) is the minimum of all symmetric p-merging functions (Theorem 3.1). Second, we give two representation results (Theorems 5.1 and 5.2) of admissible p-merging functions which are naturally connected to e-values [Vovk and Wang, 2020b, Shafer, 2019, Grünwald et al., 2020], our important technical tool, via a duality argument. Third, we provide an analytical condition for a calibrator to induce an admissible p-merging function (Theorem 6.2). Fourth, we proceed to show that the classic p-merging function of Hommel [1983] and the scaled averaging functions of Vovk and Wang [2020a] can be strictly improved to their more powerful versions (Theorems 7.1 and 8.2), whereas the scaled order statistics of Rüger [1978] are generally admissible after a trivial modification (Theorem 7.3). Various other smaller results on properties and comparisons of p-merging functions are obtained during our scientific journey.

Our p-merging functions can be directly applied to any procedures for multiple hypothesis testing, such as those of Genovese and Wasserman [2004] and Goeman and Solari [2011]; see Section 10 for simulation studies. In addition to the grid harmonic p-merging function H_K^*, strictly dominating the merging function of Hommel [1983], we design an admissible merging function $F_{-1,K}^*$ strictly dominating the harmonic merging function of Vovk and Wang [2020a]. The Hommel and harmonic merging functions have been shown to be special among two general families (see Section 4 of Chen et al. [2020]) with wide applications, attractive properties, and good empirical performance (e.g., Wilson [2020]).

Several mathematical results in this paper are quite sophisticated and surprising. In Theorem 7.1, we find the unexpected result that H_K^*, while admissible for non-prime numbers K of the input p-values, is not admissible in general for prime K. For a given p-merging function, it is generally difficult to prove or disprove its admissibility, or to construct a dominating admissible p-merging function. The proofs of our results rely on recent techniques in robust risk aggregation and dependence modeling. In particular, advanced results on joint mixability in Wang and Wang [2011, 2016] play a crucial role in proving Theorem 6.2, and many other results in the paper require complicated constructions of specific dependence structure among p-variables. Some open questions are presented in concluding Section 11 for the interested reader.

Remark 1.1. A useful distinction, introduced in Good [1958], is between statistical tests in parallel and in series. In the former case the input p-values are all based on the same evidence, and we are mostly interested in this case. In testing in series the input p-values may be based on bodies of evidence that we may judge to be independent, and then the assumption of independence of p-values may be justified. More generally, one may consider sequentially dependent (or *sequential*) p-values; cf. Vovk and Wang [2020c, Section 2].

2 P-merging functions and basic properties

Without loss of generality we fix an atomless probability space $(\Omega, \mathcal{A}, Q)$ (see, e.g., Föllmer and Schied [2011, Proposition A.27] or Vovk and Wang [2020b, Appendix D]). A *p-variable* is a random variable $P : \Omega \to [0, \infty)$ satisfying

$$Q(P \le \epsilon) \le \epsilon \text{ for all } \epsilon \in (0, 1).$$

The set of all p-variables is denoted by $\mathcal{P}_Q$. Throughout, $K \ge 2$ is an integer. A *p-merging function* of K p-values is an increasing Borel function $F : [0, \infty)^K \to [0, \infty)$ such that $F(P_1, \dots, P_K) \in \mathcal{P}_Q$ whenever $P_1, \dots, P_K \in \mathcal{P}_Q$. (Notice that the joint distribution of $P_1, \dots, P_K \in \mathcal{P}_Q$ can be arbitrary.) A p-merging function F is *symmetric* if $F(\mathbf{p})$ is invariant under any permutation of $\mathbf{p}$, and it is *homogeneous* if $F(\lambda \mathbf{p}) = \lambda F(\mathbf{p})$ for all $\lambda \in (0, 1]$ and $\mathbf{p}$ with $F(\mathbf{p}) \le 1$. All p-merging functions that we encounter in this paper are homogeneous and symmetric. Although we allow the domain of F to be $[0, \infty)^K$ in order to simplify presentation, the informative part of F is its restriction to $[0, 1]^K$. Throughout,

$\mathbf{0}$ is the K-vector of zeros, $\mathbf{1}$ is the K-vector of ones, and all vector inequalities and the operation $\wedge$ of taking the minimum of two vectors are component-wise. For $a, b, x, y \in \mathbb{R}$, $ax \wedge by$ should be understood as $(ax) \wedge (by)$.

We say that a p-merging function F *dominates* a p-merging function G if $F \le G$. The domination is *strict* if, in addition, $F(\mathbf{p}) < G(\mathbf{p})$ for at least one $\mathbf{p}$. We say that a p-merging function is *admissible* if it is not strictly dominated by any p-merging function. Analogously, we can define admissibility within smaller classes of p-merging functions, such as the class of symmetric p-merging functions. Finally, a p-merging function F is said to be *precise* if

$$\sup_{\mathbf{P} \in \mathcal{P}_Q^K} Q(F(\mathbf{P}) \le \epsilon) = \epsilon \text{ for all } \epsilon \in (0,1).$$

In other words, ϵ by ϵ, F attains the largest possible probability allowed for $F(\mathbf{P})$ to be a p-value. Precise p-merging functions are the main object studied by Vovk and Wang [2020a], where p-values are combined via averaging.

We collect some basic properties of admissible p-merging functions, which will be useful in our analysis later. In particular, an admissible p-merging function is always precise and lower semi-continuous, the limit of p-merging functions is again a p-merging function, and any p-merging function is dominated by an admissible p-merging function. The proofs of these results are put in Supplemental Article, Section A.1.

Proposition 2.1. *An admissible p-merging function is always precise.*

For an increasing Borel function $F : [0,\infty)^K \to [0,\infty)$, its lower semicontinuous version F' is given by

$$F'(\mathbf{p}) := \lim_{\lambda \uparrow 1} F(\lambda \mathbf{p}), \quad \mathbf{p} \in [0,\infty)^K. \tag{1}$$

Clearly, F' is increasing, lower semicontinuous, and $F' \le F$. Moreover, we define the *zero-one adjusted* version $\widetilde{F}$ of F by

$$\widetilde{F}(\mathbf{p}) := \begin{cases} F(\mathbf{p} \wedge \mathbf{1}) \wedge 1 & \text{if } \mathbf{p} \in (0,\infty)^K \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Proposition 2.2. *If F is a p-merging function, then both its lower semicontinuous version F' in (1) and its zero-one adjusted version $\widetilde{F}$ in (2) are p-merging functions. In particular, an admissible p-merging function is always lower semicontinuous, takes value 0 on $[0,\infty)^K \setminus (0,\infty)^K$, and satisfies $F(\mathbf{p}) = F(\mathbf{p} \wedge \mathbf{1}) \wedge 1$ for all $\mathbf{p} \in [0,\infty)^K$.*

The next result addresses the closure property of the set of p-merging functions.

Proposition 2.3. *The point-wise limit of a sequence of p-merging functions is a p-merging function.*

Combining the above results, we are able to show that any p-merging function is dominated by an admissible one.

Proposition 2.4. *Any p-merging function is dominated by an admissible p-merging function.*

Remark 2.5. Using the same proof as for Proposition 2.4, we can show that any symmetric p-merging function is dominated by a p-merging function that is admissible among symmetric p-merging functions. The same holds true if "symmetric" is replaced by "homogeneous" or "symmetric and homogeneous".

3 Some classes of p-merging functions

Similarly to Vovk and Wang [2020b], we pay special attention to two families of p-merging functions: the family based on order statistics introduced by Rüger [1978], henceforth the *O-family*, where "O" stands for "order", and the new family introduced by Vovk and Wang [2020a], henceforth the *M-family*, where "M" stands for "mean". The O-family is parameterized by $k \in \{1, \ldots, K\}$, and its kth element is the function (shown by Rüger [1978] to be a p-merging function)

$$G_{k,K} : (p_1, \ldots, p_K) \mapsto \frac{K}{k} p_{(k)} \wedge 1, \tag{3}$$

where $p_{(k)}$ is the kth order statistic of $p_1, \ldots, p_K$. The M-family is parameterized by $r \in [-\infty, \infty]$, and its element with index r has the form

$$F_{r,K} : (p_1, \ldots, p_K) \mapsto b_{r,K} M_{r,K}(p_1, \ldots, p_K) \wedge 1, \tag{4}$$

where

$$M_{r,K}(p_1, \ldots, p_K) := \left(\frac{p_1^r + \cdots + p_K^r}{K} \right)^{1/r}$$

and $b_{r,K} \geq 1$ is a suitable constant making $F_{r,K}$ a precise merging function (its value will be specified in Section 8.1). The average $M_{r,K}$ is also defined for $r \in \{0, \infty, -\infty\}$ as the limiting cases of (4), which correspond to the geometric average, the maximum, and the minimum, respectively. All members of both families are precise p-merging functions.

The initial and final elements of the M- and O-families coincide: the initial element is the Bonferroni p-merging function

$$G_{1,K} = F_{-\infty,K} : (p_1, \ldots, p_K) \mapsto K \min(p_1, \ldots, p_K) \wedge 1, \tag{5}$$

and the final element is the maximum p-merging function

$$G_{K,K} = F_{\infty,K} : (p_1, \ldots, p_K) \mapsto \max(p_1, \ldots, p_K).$$

While the Bonferroni p-merging function is constantly used in practice, the maximum p-merging function is obviously useless. For the intermediate values of k, $1 < k < K$, $G_{k,K}$ appear to be an arbitrary choice. Another prominent element of the M-family is the multiple $F_{-1,K}$ of the harmonic mean $M_{-1,K}$ [Good, 1958, Wilson, 2019], variations of which have been used in bioinformatics and other sciences. More generally, choosing a good value of r is discussed in detail in Section 6 of Vovk and Wang [2020a].

Another important p-merging function is that of Hommel [1983], given by

$$H_K := \left(\sum_{k=1}^{K} \frac{1}{k} \right) \bigwedge_{k=1}^{K} G_{k,K}.$$

To some degree it solves the problem of choosing k. The Hommel function H_K (or $H_K \wedge 1$, since a truncation at 1 is trivial) is a precise p-merging function, and it equals a constant $\ell_K := \sum_{k=1}^{K} k^{-1}$ times the function

$$S_K := \bigwedge_{k=1}^{K} G_{k,K} = \frac{1}{\ell_K} H_K,$$

used by Simes [1986]. The Simes function S_K is a valid merging function for independent p-variables (or under some other dependence assumptions, as in, e.g., Sarkar [1998]).

Admissibility of the above p-merging functions will be studied in Sections 7 and 8. In the case of inadmissibility, a function can be strictly improved to another p-merging function without losing validity (Proposition 2.4). We will explicitly construct new merging functions that strictly dominate the existing ones. In one of the two extreme special cases, the Bonferroni p-merging function is shown to be admissible in Vovk and Wang [2020b, Proposition 6.1]. On the contrary, the maximum p-merging function $G_{K,K}$ ($F_{\infty,K}$) is not admissible for any $K \geq 2$, since it is strictly dominated by, for instance, $(p_1, \dots, p_K) \mapsto p_1$. Nevertheless, after a trivial modification, $G_{K,K}$ is admissible within the class of symmetric p-merging functions; see Theorem 7.3 in Section 7.

Next, we present a result showing that the Simes function S_K has a very special role in the context of p-merging, as it is a lower bound for any symmetric p-merging functions. Therefore, $S_K(p_1, \dots, p_K)$ can be seen as the best achievable p-value obtained via symmetric merging of $p_1, \dots, p_K$, although the function S_K itself is not a valid p-merging function.

Theorem 3.1. *The Simes function S_K is the minimum of all symmetric p-merging functions.*

Proof. Take any symmetric p-merging function F and $\mathbf{p} = (p_1, \dots, p_K)$. Let $\alpha := S_K(\mathbf{p})/K$. Note that $K\alpha \leq 1$ and $p_{(k)} \geq k\alpha$ for each $k = 1, \dots, K$. By the symmetry of F,

$$F(\mathbf{p}) = F(p_{(1)}, \dots, p_{(K)}) \geq F(\alpha, 2\alpha, \dots, K\alpha) =: \beta.$$

Let Π be the set of all permutations of the vector $(\alpha, 2\alpha, \dots, K\alpha)$, and μ be the discrete uniform distribution over Π. Take a random vector $(P_1, \dots, P_K)$ following the distribution $K\alpha\mu + (1-K\alpha)\delta_{(1,\dots,1)}$. For each k, the distribution of P_k is given by $\sum_{k=1}^K \alpha\delta_{k\alpha} + (1-K\alpha)\delta_1$, and hence P_k is a p-variable. Since F is a p-merging function, we have

$$\beta \geq Q(F(P_1, \dots, P_K) \leq \beta) \geq Q((P_1, \dots, P_K) \in \Pi) = K\alpha.$$

This implies $F(\mathbf{p}) \geq K\alpha = S_K(\mathbf{p})$, and hence S_K dominates all symmetric p-merging functions. Finally, the statement of S_K as a minimum follows from $S_K = \bigwedge_{k=1}^K G_{k,K}$, noting that each $G_{k,K}$ is a symmetric p-merging function. □

In the main part of the paper we will focus on the case $K > 2$. The case $K = 2$ is very different but simpler; it is treated separately in Supplemental Article, Section B. In this case, the Bonferroni p-merging function $(p_1, p_2) \mapsto \min(2p_1, 2p_2, 1)$ is the only admissible symmetric p-merging function.

4 Duality and p-to-e merging

As a prelude to studying the problem of merging p-values into a p-value, we will discuss the notion of e-values and the much easier problem of merging p-values into an e-value [Vovk and Wang, 2020b, Appendix G]. As already mentioned, in this paper we are only interested in e-values as a technical tool.

An *e-variable* is a non-negative extended random variable $E : \Omega \to [0, \infty]$ with $\mathbb{E}^Q[E] \leq 1$. A *calibrator* (or, more fully, "p-to-e calibrator") is a decreasing function $f : [0, \infty) \to [0, \infty]$ satisfying $f = 0$ on $(1, \infty)$ and $\int_0^1 f(x)\,\mathrm{d}x \leq 1$. A calibrator transforms any p-variable to an e-variable. It is *admissible* if it is upper semicontinuous, $f(0)=\infty$, and $\int_0^1 f(x)\,\mathrm{d}x = 1$ (equivalently [Vovk and Wang, 2020b, Propositions 2.1 and 2.2], it is not strictly dominated, in a natural sense, by any other calibrator).

A function $F : [0,\infty)^K \to [0,\infty]$ is a *p-to-e merging function* if $F(P_1,\dots,P_K)$ is an e-variable for any p-variables $P_1,\dots,P_K$. A p-to-e merging function F *dominates* a p-to-e-merging function G if $F \ge G$, and the domination is *strict* if $F \ne G$; F is *admissible* if it is not strictly dominated by any other p-to-e merging function.

Below, Δ_K is the standard K-simplex, that is, $\Delta_K := \{(\lambda_1,\dots,\lambda_K) \in [0,1]^K : \lambda_1 + \cdots + \lambda_K = 1\}$, and we always write $\mathbf{p} := (p_1,\dots,p_K)$.

It is clear that a convex mixture of e-variables is an e-variable. In this sense a convex mixture is an "e-merging function"; and in the symmetric case, the arithmetic average essentially dominates any other e-merging function [Vovk and Wang, 2020b, Proposition 3.1]. Therefore, for any calibrators $f_1,\dots,f_K$ and any $(\lambda_1,\dots,\lambda_K) \in \Delta_K$, the function

$$G(\mathbf{p}) := \lambda_1 f_1(p_1) + \cdots + \lambda_K f_K(p_K) \tag{6}$$

is a p-to-e merging function.

The following corollary of a duality theorem for optimal transport says that this procedure of p-to-e merging is general.

Proposition 4.1. *For any calibrators $f_1,\dots,f_K$ and any $(\lambda_1,\dots,\lambda_K) \in \Delta_K$, (6) is a p-to-e merging function. Conversely, any p-to-e merging function F is dominated by the p-to-e merging function (6) for some calibrators $f_1,\dots,f_K$ and some $(\lambda_1,\dots,\lambda_K) \in \Delta_K$.*

Proof. The non-trivial statement is the second one. Let F be a p-to-e merging function. Denote by $\mathcal{F}$ the set of decreasing real functions on $[0,\infty)$, and define the operator $\bigoplus$ as

$$\left(\bigoplus_{k=1}^K g_k\right)(x_1,\dots,x_K) := \sum_{k=1}^K g_k(x_k), \quad (g_1,\dots,g_K) \in \mathcal{F}^K, \quad (x_1,\dots,x_K) \in [0,\infty)^K.$$

Using a classic duality theorem (see, e.g., Rüschendorf [2013, Theorem 2.3]), we have

$$\min\left\{\sum_{k=1}^K \int_0^1 g_k(x)\,\mathrm{d}x : (g_1,\dots,g_K) \in \mathcal{F}^K,\ \bigoplus_{k=1}^K g_k \ge F\right\} = \sup_{\mathbf{P}\in\mathcal{P}_Q^K} \mathbb{E}^Q[F(\mathbf{P})] \le 1. \tag{7}$$

Indeed, part (a) of Theorem 2.3 in Rüschendorf [2013] gives the equality with inf in place of min and with $\mathbf{P}$ ranging over the probability measures on $[0,1]^K$ with the uniform marginals. Part (d) of that theorem gives inf, and it remains to notice that every p-variable P is dominated, in the sense of $U \le P$, by a random variable U (perhaps on an extended probability space) uniformly distributed on $[0,1]$ (see, e.g., Rüschendorf [2009, Theorem 2.3]).

Choose $g_1,\dots,g_K$ at which the minimum is attained in (7). It is clear that we can define calibrators $f_1,\dots,f_K$ and $(\lambda_1,\dots,\lambda_K) \in \Delta_K$ in such a way that $\lambda_k f_k \ge g_k$ for all k, e.g., $\lambda_k := \int_0^1 g_k(x)\,\mathrm{d}x / \sum_{i=1}^K \int_0^1 g_i(x)\,\mathrm{d}x$ and $f_k := g_k / \int_0^1 g_k(x)\,\mathrm{d}x$ (the simple cases where one or both of the denominators vanish should be considered separately). With this choice F will be dominated by the p-to-e merging function (6). □

By the Markov inequality, $1/F$ is a p-merging function for any p-to-e merging function F. Such a "naive procedure" for merging p-values is generally not admissible. Nevertheless, for a fixed $\epsilon \in (0,1)$ and any admissible p-merging function G, we can find a p-to-e merging function F such that $G \le \epsilon \Leftrightarrow F \ge 1/\epsilon$. These statements are discussed and put in a more general context in Section A.5 of Supplemental Article.

5 Rejection regions of admissible p-merging functions

A p-merging function can be characterized by its rejection regions. The *rejection region* of a p-merging function F at level $\epsilon > 0$ is defined as

$$R_\epsilon(F) := \left\{ \mathbf{p} \in [0,\infty)^K : F(\mathbf{p}) \le \epsilon \right\}. \tag{8}$$

If F is homogeneous, then $R_\epsilon(F)$, $\epsilon \in (0,1)$, takes the form $R_\epsilon(F) = \epsilon A$ for some $A \subseteq [0,\infty)^K$.

Conversely, any increasing collection of Borel lower sets $\{R_\epsilon \subseteq [0,\infty)^K : \epsilon \in (0,1)\}$ determines an increasing Borel function $F : [0,\infty)^K \to [0,1]$ by the equation

$$F(\mathbf{p}) = \inf\{\epsilon \in (0,1) : \mathbf{p} \in R_\epsilon\}, \tag{9}$$

with the convention $\inf \varnothing = 1$. It is immediate that F is a p-merging function if and only if $Q(\mathbf{P} \in R_\epsilon) \le \epsilon$ for all $\epsilon \in (0,1)$ and $\mathbf{P} \in \mathcal{P}_Q^K$.

The main result in this section is a representation of rejection regions of admissible p-merging functions. It turns out that calibrating p-values into e-values as in Proposition 4.1 is a useful technical tool for studying such rejection regions.

Theorem 5.1. *For any admissible homogeneous p-merging function F, there exist $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and admissible calibrators $f_1, \dots, f_K$ such that*

$$R_\epsilon(F) = \epsilon \left\{ \mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \lambda_k f_k(p_k) \ge 1 \right\} \qquad \text{for each } \epsilon \in (0,1). \tag{10}$$

Conversely, for any $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and calibrators $f_1, \dots, f_K$, (10) determines a homogeneous p-merging function.

Proof. Fix an arbitrary $\epsilon \in (0,1)$. Note that the set $R_\epsilon(F)$ is a lower set, and it is closed due to Proposition 2.2. We use the same notation as in the proof of Proposition 4.1. Using the duality relation (7),

$$\min_{(g_1,\dots,g_K)\in\mathcal{F}^K} \left\{ \sum_{k=1}^K \int_0^1 g_k(x)\,\mathrm{d}x : \bigoplus_{k=1}^K g_k \ge 1_{R_\epsilon(F)} \right\} = \max_{\mathbf{P}\in\mathcal{P}_Q^K} Q(\mathbf{P} \in R_\epsilon(F)) = \epsilon,$$

where the last equality holds because F is precise (Proposition 2.1). Take $(g_1^\epsilon, \dots, g_K^\epsilon) \in \mathcal{F}^K$ such that $\bigoplus_{k=1}^K g_k^\epsilon \ge 1_{R_\epsilon(F)}$ and $\sum_{k=1}^K \int_0^1 g_k^\epsilon(x)\,\mathrm{d}x = \epsilon$. Obviously we can choose each g_k^ϵ to be non-negative and left-continuous. Using the fact that $R_\epsilon(F)$ is a closed lower set, we have

$$\max_{\mathbf{P}\in\mathcal{P}_Q^K} Q(\mathbf{P} \in R_\epsilon(F)) = \epsilon \implies \max_{\mathbf{P}\in\mathcal{P}_Q^K} Q(\epsilon\mathbf{P} \in R_\epsilon(F)) = 1. \tag{11}$$

Therefore, using duality again,

$$\min_{(g_1,\dots,g_K)\in\mathcal{F}^K} \left\{ \sum_{k=1}^K \frac{1}{\epsilon} \int_0^\epsilon g_k(x)\,\mathrm{d}x : \bigoplus_{k=1}^K g_k \ge 1_{R_\epsilon(F)} \right\} = 1,$$

implying $\sum_{k=1}^K \int_0^\epsilon g_k^\epsilon(x)\,\mathrm{d}x \ge \epsilon$. As $g_k \ge 0$ for each k and $\sum_{k=1}^K \int_0^1 g_k^\epsilon(x)\,\mathrm{d}x = \epsilon$, we know $g_k^\epsilon(x) = 0$ for $x > \epsilon$.

Define the set $A_\epsilon := \{\mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K g_k^\epsilon(p_k) \geq 1\}$. Since $\bigoplus_{k=1}^K g_k^\epsilon \geq 1_{R_\epsilon(F)}$, we have $R_\epsilon(F) \subseteq A_\epsilon$. Note that A_ϵ is a closed lower set. By Markov's inequality,

$$\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q\left(\bigoplus_{k=1}^K g_k^\epsilon(\mathbf{P}) \geq 1\right) \leq \sup_{P\in\mathcal{P}_Q} \sum_{k=1}^K \mathbb{E}^Q[g_k^\epsilon(P)] = \epsilon.$$

Hence, we can define a function $F' : [0,\infty)^K \to \mathbb{R}$ via $R_\epsilon(F') = A_\epsilon$ and $R_\delta(F') = \delta\epsilon^{-1}A_\epsilon$ for all $\delta \in (0,1)$. By the above properties of A_ϵ, F' is a valid homogeneous p-merging function. Moreover, F' dominates F since $R_\delta(F) \subseteq A_\delta$ for all $\delta \in (0,1)$ due to homogeneity of F. The admissibility of F now gives $F = F'$, and thus

$$R_\epsilon(F) = A_\epsilon = \epsilon\left\{\mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K g_k^\epsilon(\epsilon p_k) \geq 1\right\} \qquad \text{for each } \epsilon \in (0,1).$$

Note that $A := \epsilon^{-1}R_\epsilon(F) = \epsilon^{-1}A_\epsilon$ does not depend on $\epsilon \in (0,1)$. For a fixed $\epsilon \in (0,1)$, let $\lambda_k := \epsilon^{-1}\int_0^\epsilon g^\epsilon(x)\,\mathrm{d}x$ and $f_k : (0,\infty) \to \mathbb{R}$, $x \mapsto g_k^\epsilon(\epsilon x)/\lambda_k$ for each $k = 1,\dots,K$ (if $\lambda_k = 0$, then let $f_k := 1$), and further set $f_k(0) = \infty$. It is clear that for each k with $\lambda_k \neq 0$,

$$\int_0^1 f_k(x)\,\mathrm{d}x = \frac{\int_0^1 \epsilon g_k^\epsilon(\epsilon x)\,\mathrm{d}x}{\int_0^1 g_k^\epsilon(x)\,\mathrm{d}x} = \frac{\int_0^\epsilon g_k^\epsilon(x)\,\mathrm{d}x}{\int_0^1 g_k^\epsilon(x)\,\mathrm{d}x} = 1.$$

The conditions that f_k is decreasing and left-continuous, $\int_0^1 f_k(x)\,\mathrm{d}x = 1$, $f_k(0) = \infty$, and $f_k(x) = 0$ for $x > 1$ imply that f_k is an admissible calibrator. Therefore, (10) holds.

For the last statement, let $f_1,\dots,f_K$ be calibrators and $(\lambda_1,\dots,\lambda_K) \in \Delta_K$. Note that for each $\epsilon \in (0,1)$, (10) gives

$$R_\epsilon(F) = \left\{\mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \lambda_k f_k\left(\frac{p_k}{\epsilon}\right) \geq 1\right\},$$

and since $f(x) = 0$ for $x > 1$, it holds

$$\sum_{k=1}^K \lambda_k \int_0^1 f_k\left(\frac{x}{\epsilon}\right)\,\mathrm{d}x = \sum_{k=1}^K \lambda_k \int_0^{1/\epsilon} f_k(y)\,\mathrm{d}y = \epsilon\sum_{k=1}^K \lambda_k = \epsilon.$$

Hence, Markov's inequality gives

$$\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q\left(\mathbf{P} \in R_\epsilon(F)\right) = \sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q\left(\bigoplus_{k=1}^K \lambda_k f_k\left(\frac{\mathbf{P}}{\epsilon}\right) \geq 1\right) \leq \epsilon.$$

Thus, (10) determines a homogeneous p-merging function. $\square$

As an immediate consequence of (10), for an admissible homogeneous p-merging function F and $\epsilon \in (0,1)$, $F(p_1,\dots,p_K) \leq \epsilon$ if and only if $F(p_1 \wedge \epsilon,\dots,p_K \wedge \epsilon) \leq \epsilon$. Therefore, for a rejection region of F at level ϵ, there is no dependence on input p-values larger than ϵ.

If the homogeneous p-merging function F is symmetric, then $f_1,\dots,f_K$, as well as $\lambda_1,\dots,\lambda_K$, in Theorem 5.1 can be chosen identical.

Theorem 5.2. *For any F that is admissible within the family of homogeneous symmetric p-merging functions, there exists an admissible calibrator f such that*

$$R_\epsilon(F) = \epsilon \left\{ \mathbf{p} \in [0,\infty)^K : \frac{1}{K}\sum_{k=1}^{K} f(p_k) \geq 1 \right\} \qquad \textit{for each } \epsilon \in (0,1). \tag{12}$$

Conversely, for any calibrator f, (12) determines a homogeneous symmetric p-merging function.

Proof. The proof is similar to that of Theorem 5.1 and we only mention the differences. For the first statement, it suffices to notice two facts. First, if R_ϵ is symmetric, then $g_1^\epsilon, \dots, g_K^\epsilon$ in the proof of Theorem 5.1 can be chosen as identical; for instance, one can choose the average of them (see, e.g., Proposition 2.5 of Rüschendorf [2013]). Second, the symmetry of $R_\epsilon(F)$ guarantees that F' in the proof of Theorem 5.1 is symmetric, and hence it is sufficient to require the admissibility of F within homogeneous symmetric p-merging functions in this proposition. The last statement in the proposition follows from Theorem 5.1 by noting that (12) defines a symmetric rejection region. □

Remark 5.3. In the converse statements of Theorems 5.1 and 5.2, a p-merging function induced by admissible calibrators is not necessarily admissible (see Example 5.5), although admissibility is indispensable in the proof of the forward direction. Using (11) and a compactness argument, a necessary and sufficient condition for a calibrator f to induce a precise p-merging function (a weaker requirement than admissibility) via (12) is

$$Q\left(\frac{1}{K}\sum_{k=1}^{K} f(P_k) = 1\right) = 1 \quad \text{for some } P_1, \dots, P_K \sim \mathrm{U}[0,1]. \tag{13}$$

Condition (13) may be difficult to check for a given f in general. For a convex f, as shown by Wang and Wang [2011, Theorem 2.4], (13) holds if and only if $f \leq K$ on $(0,1]$. Sufficient conditions for admissibility will be studied in Section 6 below. Similarly to (13), an equivalent condition for the p-merging function F in (10) to be precise is

$$Q\left(\sum_{k=1}^{K} \lambda_k f_k(P_k) = 1\right) = 1 \quad \text{for some } P_1, \dots, P_K \sim \mathrm{U}[0,1]. \tag{14}$$

Using the terminology of Wang and Wang [2016], (14) means that the distributions of $\lambda_k f_k(P_k)$, $k = 1, \dots, K$, are jointly mixable. Assuming convexity of the calibrators, (14) has a similar equivalent condition [Wang and Wang, 2016, Theorem 3.2], and this result is essential to the proof of Theorem 6.2 below.

For a decreasing function $f : [0,\infty) \to [0,\infty]$ and a p-merging function F taking values in $[0,1]$, we say that f *induces* F if (12) holds; similarly, we say that $\lambda_1, \dots, \lambda_K$ and $f_1, \dots, f_K$ *induce* F if (10) holds. Theorems 5.1 and 5.2 imply that admissible p-merging functions are induced by some admissible calibrators. Generally, the calibrator inducing a given p-merging function may not be unique. In the following examples, p-merging functions are induced by calibrators, although these p-merging functions are not necessarily admissible.

Example 5.4. The p-merging function $F := G_{k,K}$, $k \in \{1, \dots, K\}$, is induced by the calibrator $(K/k)1_{[0,k/K]}$.

Example 5.5. In the case $K = 2$, the p-merging function

$$F : \mathbf{p} \mapsto 2M_{1,K}(\mathbf{p} \wedge \mathbf{1}) \wedge 1_{\{\min \mathbf{p}>0\}} = 2M_{1,K}(\mathbf{p}) \wedge 1_{\{\min \mathbf{p}>0\}}$$

is induced by the admissible calibrator $f : x \mapsto (2 - 2x)_+$ on $(0, \infty)$ and $f(0) = \infty$. The function F is the zero-one adjusted version (see Proposition 2.2) of the arithmetic merging function, and it is dominated by the Bonferroni merging function. Hence, F is not admissible.

Example 5.6. One may also generate p-merging functions from (12) where f is not a calibrator. For the arithmetic merging function $F := 2M_{1,K}$, equality (12) holds by choosing the function $f : x \mapsto 2 - 2x$. Note that f is not a calibrator and it takes negative values for $x > 1$. For another example, we take $F := F_{r,K}$ for $r < 0$ in (4). Rewriting the equation $F(\epsilon\mathbf{p}) \leq \epsilon$ as $b_{r,K}(\frac{1}{K}\sum_{k=1}^{K} p_k^r)^{1/r} \leq 1$, we see that

$$R_\epsilon(F) = \epsilon \left\{ \mathbf{p} \in [0, \infty)^K : \frac{1}{K}\sum_{k=1}^{K} b_{r,K}^r p_k^r \geq 1 \right\},$$

thus satisfying (12) with $f : x \mapsto b_{r,K}^r x^r$. Such f is generally not a calibrator (not even integrable for $r \leq -1$), although it induces a precise p-merging function for a properly specified value of $b_{r,K}$ in Section 8.

The requirement $f(0) = \infty$ for an admissible calibrator f implies that the combined test (12) gives a rejection as soon as one of the input p-values is 0, which is obviously necessary for admissibility (Proposition 2.2). Although many examples in the M- and O-families, in particular $F_{r,K}$ for $r > 0$ and $G_{k,K}$ for $k > 1$, do not satisfy this, we can make the zero-one adjustment (2), which does not affect the validity of the p-merging function by Proposition 2.2. In the sequel, a calibrator will be specified by its values on $(0, 1]$, as $f = 0$ on $(1, \infty)$ for any calibrator f, and $f(0)$ should be clear in each specific example (in particular $f(0) = \infty$ if f is admissible). The value $f(0)$ does not affect the p-merging function determined by (12) as long as $f(0) \geq K$.

6 Conditions for admissibility

We have seen that p-merging functions induced by admissible calibrators via Theorems 5.1 and 5.2 are not necessarily admissible (Example 5.5). In this section, we study sufficient conditions for admissibility based on calibrators. First, Theorems 5.1 and 5.2 lead to an immediate criterion for checking the admissibility of an induced p-merging function (proved in Section A.2 of Supplemental Article).

Proposition 6.1. *Suppose that F is a p-merging function taking values in $[0, 1]$ and satisfying (12) for a decreasing function f. The following statements hold:*

(i) F is admissible among symmetric p-merging functions if and only if there is no calibrator g such that

$$\left\{ \mathbf{p} \in [0, \infty)^K : \frac{1}{K}\sum_{k=1}^{K} f(p_k) \geq 1 \right\} \subsetneq \left\{ \mathbf{p} \in [0, \infty)^K : \frac{1}{K}\sum_{k=1}^{K} g(p_k) \geq 1 \right\}. \qquad (15)$$

(ii) F is admissible if and only if there are no $(\lambda_1,\dots,\lambda_K)\in\Delta_K$ and calibrators $g_1,\dots,g_K$ such that

$$\left\{\mathbf{p}\in[0,\infty)^K : \frac{1}{K}\sum_{k=1}^K f(p_k)\ge 1\right\} \subsetneq \left\{\mathbf{p}\in[0,\infty)^K : \sum_{k=1}^K \lambda_k g_k(p_k)\ge 1\right\}. \tag{16}$$

Note that (15) does not imply $g\ge f$, making the existence of g often complicated to analyze. Proposition 6.1 implies, in particular, that for any calibrator f, $f\le K$ on $(0,1]$ is a necessary condition for the induced p-merging function to be admissible, because otherwise the function $g: x\mapsto f(cx)\wedge K$ where $c:=\int_0^1 f(x)\wedge K\,\mathrm{d}x<1$ would induce a p-merging function strictly dominating F. On the other hand, if $f(1)>0$, then the calibrator $g:=(f-f(1))/(1-f(1))1_{[0,1]}$ induces the same p-merging function F. Hence, it suffices to consider f with $f\le K$ on $(0,1]$ and $f(1)=0$.

The main result of this section gives a sufficient condition for the admissibility of the corresponding p-merging function. For a calibrator f, we define another calibrator $g: [0,\infty)\to[0,\infty]$, for some $\eta\in[0,1/K]$, via

$$g: x\mapsto f\left(\frac{x-\eta}{1-K\eta}\right)1_{\{x\in(\eta,1-(K-1)\eta]\}} + K1_{\{x\in[0,\eta]\}}. \tag{17}$$

It is straightforward to verify $\int_0^1 g(x)\,\mathrm{d}x\le 1$, and g defined via (17) is a calibrator.

Theorem 6.2. *Suppose that an admissible calibrator f is strictly convex or strictly concave on $(0,1]$, $f(0+)\in(K/(K-1),K]$, and $f(1)=0$. The p-merging function induced by f, or g in (17) for any $\eta\in[0,1/K]$, is admissible.*

Proof. We will prove the statement on f, and the statement on g would then follow from Lemma A.1 in Supplemental Article, Section A.2, which says that if f induces an admissible p-merging function, then so does g in (17). We only show the case where f is strictly convex, as the case of a strictly concave f follows from a symmetric argument; we remark that $f(0+)\le K$ for a convex f and $f(0+)>K/(K-1)$ for a concave f play the same role in the proof.

Suppose for the purpose of contradiction that there exists a p-merging function G which strictly dominates F, that is, there exist $\mathbf{p}=(p_1,\dots,p_K)\in[0,1]^K$ and $\alpha\in(0,1)$ such that $G(\mathbf{p})<\alpha<F(\mathbf{p})<1$. Set $a:=\lim_{t\downarrow 0} f(t)\le K$. Clearly, $a>2$ since no strictly convex function on $[0,1]$ bounded by 2 integrates to 1. Hence, it suffices to assume $K\ge 3$.

Note that f is continuous and strictly decreasing on $(0,1)$. Let $f^{-1}:(0,a)\mapsto(0,1)$ be the inverse function of f, which is strictly decreasing and strictly convex. Let U be a uniform random variable on $[0,1]$, and let h be the density function $f(U)$. Note that h is a strictly decreasing density function. Since $\mathbf{p}\notin R_\alpha(F)$, we have $\sum_{k=1}^K f(p_k/\alpha)<K$. Denote by $y_k:=f(p_k/\alpha)$, $k=1,\dots,K$. Note that $y_1+\dots+y_K<K$ and $y_k<a$ for each k. Take a small constant

$$\epsilon := \frac{1}{4}\min\left\{\bigwedge_{k=1}^K (a-y_k),\ a-2,\ 1-\frac{1}{K}\sum_{k=1}^K y_k\right\}>0.$$

For each $k=1,\dots,K$, h is strictly decreasing in $[y_k+\epsilon, y_k+2\epsilon]$ since $y_k+2\epsilon\le a-2\epsilon$. Define another density function $v_k:=(h-h(y_k+2\epsilon))1_{[y_k+\epsilon,y_k+2\epsilon]}$ with its mass $m_k:=\int_{y_k+\epsilon}^{y_k+2\epsilon} v_k(t)\,\mathrm{d}t>0$ and its mean $\mu(v_k)$ smaller than $y_k+2\epsilon$.

Write $\beta := 1-\frac{1}{K}(\mu(v_1)+\cdots+\mu(v_K))$. Since $\mu(v_1)+\cdots+\mu(v_K) < y_1+\cdots+y_K+2K\epsilon < K$, we have $\beta > 0$. Take another small constant

$$\theta := \min\left\{\bigwedge_{k=1}^{K}\frac{m_k\beta}{a-1},\ f^{-1}(a-\epsilon),\ \frac{(1-\alpha)(K-1)}{\alpha}\right\} > 0,$$

and let

$$m^* := \frac{\int_0^\theta f(t)\mathrm{d}t - \theta}{\beta} \le \frac{(a-1)\theta}{\beta} \le \bigwedge_{k=1}^{K} m_k.$$

We have $\int_\theta^1 f(t)\,\mathrm{d}t = 1-\int_0^\theta f(t)\,\mathrm{d}t = 1-\theta-m^*\beta$. Note that $a > f(\theta) \ge a-\epsilon > \bigvee_{k=1}^{K} y_k+2\epsilon$. For $k=1,\dots,K$, define a probability density function

$$h_k = \frac{1}{1-\theta-m^*}\left(h1_{(0,f(\theta)]} - m^*\frac{v_k}{m_k}\right), \tag{18}$$

which is supported in interval $(0, f(\theta)]$, and its mean $\mu(h_k)$ satisfies

$$\mu(h_k) = \frac{\int_\theta^1 f(t)\,\mathrm{d}t - m^*\mu(v_k)}{1-\theta-m^*} = \frac{1-\theta-m^*\beta-m^*\mu(v_k)}{1-\theta-m^*}.$$

We have

$$\sum_{k=1}^{K}\mu(h_k) = \frac{K(1-\theta-m^*\beta)-m^*\sum_{k=1}^{K}\mu(v_k)}{1-\theta-m^*} = K > f(\theta).$$

Note that each of $h_1,\dots,h_K$ has a decreasing density in $(0, f(\theta)]$, and the sum of their means is larger than $f(\theta)$, thus satisfying the condition of joint mixability in Wang and Wang [2016, Theorem 3.2]. Using that theorem, there exists a random vector $\mathbf{X} = (X_1,\dots,X_K)$ satisfying $X_k \sim h_k$, $k=1,\dots,K$, and $X_1+\cdots+X_K = K$.

Take disjoint events $A, B, C, B_1,\dots,B_K$ independent of $\mathbf{X}$ such that $Q(A) = (1-\theta-m^*)\alpha$, $Q(B) = m^*\alpha$, $Q(C) = 1-\alpha-\theta\alpha/(K-1)$ and $Q(B_1) = \cdots = Q(B_K) = \theta\alpha/(K-1)$. Design a random vector $\mathbf{P} = (P_1,\dots,P_K)$ by letting, for $k=1,\dots,K$,

$$P_k = \alpha f^{-1}(X_k)1_A + p_k 1_B + \sum_{j=1, j\neq k}^{K}\theta\alpha 1_{B_j} + 1_{B_k} + 1_C. \tag{19}$$

The decomposition (18) gives, for each $k=1,\dots,K$, that

$$\frac{Q(f^{-1}(X_k)1_A + f^{-1}(y_k)1_B > x)}{(1-\theta)\alpha} \ge \frac{1-x}{1-\theta} \qquad \text{for all } x\in(\theta,1),$$

and thus the conditional distribution of $f^{-1}(X_k)1_A + f^{-1}(y_k)1_B$ on $A\cup B$ is stochastically larger than the $\mathrm{U}[\theta,1]$. As a consequence, the distribution of P_k is stochastically larger than $\theta\alpha\delta_{\theta\alpha} + (1-\theta)\alpha\mathrm{U}[\theta\alpha,\alpha] + (1-\alpha)\delta_1$, and hence P_k is a p-variable.

If A happens, then $f(P_k/\alpha) = X_k$ for each k, and $\sum_{k=1}^{K} f(P_k/\alpha) = \sum_{k=1}^{K} X_k = K$. If any of B_k happens, then $\sum_{k=1}^{K} f(P_k/\alpha) = (K-1)f(\theta) > (K-1)(a-\epsilon) > K$. In both cases, using (12), $\mathbf{P}\in R_\alpha(F)\subseteq R_\alpha(G)$. If B happens, then $\mathbf{P} = \mathbf{p}\in R_\alpha(G)$. Therefore,

$$Q(\mathbf{P}\in R_\alpha(G)) \ge Q(A) + Q(B) + \sum_{k=1}^{K} Q(B_k) = \alpha + \frac{\theta\alpha}{K-1} > \alpha, \tag{20}$$

a contradiction to G being a p-merging function. This shows that F is admissible. □

Algorithm 1 The p-merging function induced by a calibrator f to accuracy 2^{-M}

Require: A calibrator f, $M \in \mathbb{N}$, and a sequence of p-values $p_1, \dots, p_K$.

$L := 0$ and $R := 1$

for $m = 1, \dots, M$ **do**

  $\epsilon := (L + R)/2$

  if $\frac{1}{K} \sum_{k=1}^{K} f(p_k/\epsilon) \geq 1$ **then** $R := \epsilon$ **else** $L := \epsilon$

return R

Rephrasing the condition on g in Theorem 6.2, we get a sufficient condition on an admissible calibrator f to ensure that the induced p-merging function is admissible:

$$\begin{aligned}&\text{For some } \eta \in [0, \tfrac{1}{K}) \text{ and } \tau := 1 - (K-1)\eta\text{: } f = K \text{ on } (0, \eta],\ f(\eta+) \in (\tfrac{K}{K-1}, K], \\ &f \text{ is strictly convex or strictly concave on } (\eta, \tau], \text{ and } f(1) = 0.\end{aligned} \tag{21}$$

Notice that the condition $f(\eta+) \in (\frac{K}{K-1}, K]$ in (21) and Theorem 6.2 excludes the simple case $K = 2$ (treated in Supplemental Article, Section B). One may try to relax the requirement that convexity or concavity be strict; we explain technical difficulties in Remark A.6 in Supplemental Article, Section A.6, for the interested reader.

A natural way to compute the p-merging function induced by a calibrator f to accuracy 2^{-M}, where M is a natural number, is to use binary search, which is given as Algorithm 1. The value of this merging function is given by $\phi_{\mathbf{p}}^{-1}(1) \wedge 1$, where $\phi_{\mathbf{p}}^{-1}$ is the left-inverse of

$$\phi_{\mathbf{p}} : \epsilon \mapsto \frac{1}{K} \sum_{k=1}^{K} f(p_k/\epsilon),$$

and the algorithm essentially solves the equation $\phi_{\mathbf{p}}(\epsilon) = 1$. Assuming that the calibrator f is computable in time $O(1)$, merging K p-values by Algorithm 1 takes time $O(MK)$. Notice that Algorithm 1 always produces a valid p-value (which exceeds the p-value produced by the p-merging function induced by f by at most 2^{-M}).

In the following few sections, we analyze admissibility of the Hommel function, members of the O-family, and members of the M-family. In cases of non-admissibility, we construct a dominating admissible p-merging function. It turns out that, except for the Bonferroni p-merging function, none of these p-merging functions has a calibrator satisfying the condition (21), and many of them can indeed be improved, either trivially or significantly. Theorem 6.2 becomes very useful in the construction of admissible p-merging functions dominating the ones in the M-family.

7 Hommel's function and the O-family

This section is dedicated to the admissibility of the Hommel function H_K and the O-family of p-merging functions $(G_{k,K})_{k=1,\dots,K}$ for a given K. The calibrators we see below are generally not continuous, and hence they do not satisfy the condition in Theorem 6.2. Nevertheless, some alternative arguments will justify the (in-)admissibility of the induced functions. The key result of this section is Theorem 7.1 about the grid harmonic p-merging function.

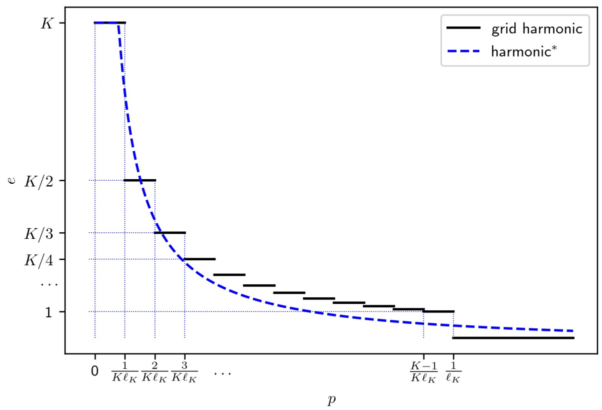

Figure 1: The grid harmonic calibrator (solid and black) and the harmonic* calibrator (dashed and blue), for $K := 12$

7.1 Grid harmonic merging function

We first show that the Hommel function $H_K \wedge 1$ is not admissible, and it can be strictly improved to an admissible p-merging function H_K^*. Recall that H_K is given by $H_K := \ell_K \bigwedge_{k=1}^K G_{k,K}$, where $\ell_K := \sum_{k=1}^K \frac{1}{k}$. Our modification H_K^* of the Hommel function will be induced by the function $f : [0,\infty) \to [0,\infty)$ defined by

$$f : x \mapsto \frac{K 1_{\{\ell_K x \le 1\}}}{\lceil K \ell_K x \rceil}, \tag{22}$$

which we call the *grid harmonic calibrator* and whose graph is shown in Figure 1 as the black piece-wise horizontal line. It is straightforward to check that f is decreasing, $f(1) = 0$, and $\int_0^1 f(x)\,\mathrm{d}x = 1$, and hence f is indeed a calibrator. We will also refer to H_K^* as the *grid harmonic p-merging function.*

Theorem 7.1. *The p-merging function $H_K \wedge 1$ is dominated (strictly if $K \ge 4$) by the grid harmonic p-merging function H_K^*. Moreover, H_K^* is always admissible among symmetric p-merging functions, and it is admissible if K is not a prime number.*

Proof. Since f induces H_K^*, by Theorem 5.2, H_K^* is a p-merging function.

Let us verify that $H_K \ge H_K^*$. The rejection region of H_K^* satisfies

$$R_\epsilon(H_K^*) = \left\{ \mathbf{p} \in [0,\infty)^K : \sum_{k=1}^K \frac{1_{\{\ell_K p_k \le \epsilon\}}}{\lceil K \ell_K p_k/\epsilon \rceil} \ge 1 \right\}. \tag{23}$$

For any $\mathbf{p} \in [0,\infty)^K$ and $\epsilon > 0$, if $H_K(\mathbf{p}) \le \epsilon$, then there exists $m = 1,\dots,K$ such that $\#\{k : K\ell_K p_k/m \le \epsilon\} \ge m$. It follows that

$$\sum_{k=1}^K \frac{1_{\{\ell_K p_k \le \epsilon\}}}{\lceil K\ell_K p_k/\epsilon \rceil} \ge \sum_{k=1}^K \frac{1}{m} 1_{\{K\ell_K p_k/\epsilon \le m\}} = \frac{1}{m} \#\{k : K\ell_K p_k/m \le \epsilon\} \ge 1.$$

By (23), $\mathbf{p} \in R_\epsilon(H_K^*)$, and thus $H_K^*(\mathbf{p}) \le \epsilon$. This shows $H_K \ge H_K^*$. It is easy to check that the reverse direction holds (i.e., $H_K = H_K^*$) if and only if $K \le 3$.

Next, we prove the admissibility of H_K^*. Set $\tau := 1/(K\ell_K)$. Using Proposition 6.1, suppose, for the purpose of contradiction, that there exists a calibrator g satisfying (15). For $x \in (0, K\tau]$, set $p_1 = \dots = p_m = x$ and $p_{m+1} = \dots = p_K > 1$, where $m := \lceil \tau x \rceil$. Since $f(x) = K/m$, we have $\sum_{k=1}^K f(p_k) = K$. Using (15), $K \le \sum_{k=1}^K g(p_k) = mg(x)$, and thus $g(x) \ge K/m = f(x)$.

Since $x \in (0, K\tau]$ is arbitrary, we have $\int_0^{K\tau} g(x)\mathrm{d}x \ge \int_0^{K\tau} f(x)\mathrm{d}x = 1$. As g is a calibrator, this means $g = f$ almost everywhere on $[0, 1]$. Moreover, f is left-continuous, which further implies $g \le f$. Hence, both sides of (15) coincide, leading to a contradiction. Thus, H_K^* is admissible among symmetric p-merging functions.

Finally, we show that H_K^* is admissible if K is not a prime number. Suppose that there exist $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and calibrators $g_1, \dots, g_K$ satisfying (16). For each $m, k = 1, \dots, K$, set $y_{m,k} := \lambda_k g_k(m\tau)$ and $T_m := \sum_{k=1}^K y_{m,k}$.

Fix any $m = 1, \dots, K$. Let Π_m be the set of all subsets of $\{1, 2, \dots, K\}$ of exactly m elements. There are $\binom{K}{m}$ elements (sets) in Π_m. For any $J \in \Pi_m$, take any $\beta > 1$ and let $\mathbf{p} = (p_1, \dots, p_K)$ be given by $p_k = m\tau 1_{\{k\in J\}} + \beta 1_{\{k\notin J\}}$, $k = 1, \dots, K$. Since $\sum_{k=1}^K f(p_k) = K$, (16) implies $1 \le \sum_{k=1}^K \lambda_k g_k(m\tau) = \sum_{k\in J} y_{m,k}$. Therefore,

$$\binom{K}{m} \le \sum_{J\in\Pi_m}\sum_{k\in J} y_{m,k} = \binom{K-1}{m-1}\sum_{k=1}^K y_{m,k} = \binom{K-1}{m-1} T_m.$$

This gives $T_m \ge K/m$.

For $x \in ((m-1)\tau, m\tau]$ and each k, we have $\lambda_k g_k(x) \ge \lambda_k g_k(m\tau) = y_{m,k}$, and hence $\lambda_k \ge \int_0^{K\tau} \lambda_k g_k(x)\mathrm{d}x \ge \tau \sum_{m=1}^K y_{m,k}$. Therefore,

$$\sum_{m=1}^K T_m = \sum_{m=1}^K\sum_{k=1}^K y_{m,k} = \sum_{k=1}^K\sum_{m=1}^K y_{m,k} \le \frac{1}{\tau}\sum_{k=1}^K \lambda_K = \frac{1}{\tau} = \sum_{m=1}^K \frac{K}{m}. \tag{24}$$

Putting $\sum_{k\in J} y_{m,k} \ge 1$, $T_m \ge K/m$ and (24) together, we get $T_m = K/m$ for each $m = 1, \dots, K$, and $\sum_{k\in J} y_{m,k} = 1$ for each $J \in \Pi_m$. This further implies $y_{m,k} = 1/m$ for all $m \le K-1$ and all k. Note that the case of $m = K$ is not concluded here since Π_K only has one element, and the analysis of this case requires K to not be a prime number. Write $K = k_1 k_2$ for some integers $k_1, k_2 \ge 2$.

Take any $I \in \Pi_{k_1}$ and $J \in \Pi_{k_2-1}$ such that $I \cap J = \varnothing$, by noting that $k_1 + k_2 - 1 < K$. Let $\mathbf{p} = (p_1, \dots, p_K)$ be given by

$$p_k = K\tau 1_{\{k\in I\}} + k_2\tau 1_{\{k\in J\}} + \beta 1_{\{k\notin I\cup J\}}, \qquad k = 1, \dots, K.$$

We have $\sum_{k=1}^K f(p_k) = k_1 + (k_2-1)K/k_2 = K$. By (16) and $y_{k_2,k} = 1/k_2$, we have

$$1 \le \sum_{k=1}^K \lambda_k g_k(p_k) = \sum_{k\in I} y_{K,k} + \sum_{k\in J} y_{k_2,k} = \sum_{k\in I} y_{K,k} + (k_2-1)\frac{1}{k_2}.$$

Hence, $\sum_{k\in I} y_{K,k} \ge k_1/K$ for any $I \in \Pi_{k_1}$. On the other hand, $\sum_{k=1}^K y_{K,k} = T_K = 1$, which leads to $y_{K,k} = 1/K$ for all $k = 1, \dots, K$. Therefore, we obtain $y_{m,k} = \frac{1}{m}$ for all

$m, k = 1, \dots, K$. This implies

$$\lambda_k \geq \int_0^{K\tau} \lambda_k g_k(x)\mathrm{d}x \geq \tau \sum_{m=1}^{K} y_{m,k} = \frac{1}{K}.$$

Since $\sum_{k=1}^{K} \lambda_k = 1$, we now know $g_k = f$ almost everywhere, which further implies $g_k \leq f$, and $\lambda_k = 1/K$, $k = 1, \dots, K$. Therefore, both sides of (16) coincide, which is a contradiction. Thus, H_K^* is admissible if K is not a prime number. □

For computing H_K^*, we can use our generic algorithm, Algorithm 1, which takes time $O(-K \log \delta)$, where δ is the desired accuracy. A precise expression is, e.g.,

$$H_K^*(p_1, \dots, p_K) := \min \left\{ \epsilon := \frac{K \ell_K p_j}{i} : i, j \in \{1, \dots, K\}, \frac{1}{K} \sum_{k=1}^{K} f\left(\frac{p_k}{\epsilon}\right) \geq 1 \right\},$$

where the range of ϵ follows from f changing its value only at the points of the form $i/(K\ell_K)$. However, this expression takes time $O(K^3)$ to compute.

Since $f(x) \leq 1/(\ell_K x)$, we have $H_K^* \geq \ell_K M_{-1,K}$. It is instructive to compare this with the row of Vovk and Wang [2020a, Table 1] for the harmonic mean.

Using Theorem 3.1, we have $S_K \leq F \leq H_K$ for any symmetric p-merging function F dominating H_K, including $F = H_K^*$. Hence, the improvement of any F over H_K, measured by the ratio H_K/F, should always be in $[1, \ell_K]$. The improvement ratio H_K/H_K^* will be analyzed in Section 9.

In Theorem 7.1, we obtain that H_K^* is admissible if K is not a prime number. Quite surprisingly, if K is a prime number, then H_K^* may be strictly dominated by some non-symmetric p-merging functions. In the following simple example, we give the dominating functions for $K = 2$ and $K = 3$. More complicated examples can be constructed for larger prime numbers, although we do not know whether K being prime always implies non-admissibility of H_K^*.

Example 7.2. In the case $K = 2$, $H_2^* : (p_1, p_2) \mapsto 3p_{(1)} \wedge \frac{3}{2}p_{(2)}$ is strictly dominated by $F : (p_1, p_2) \mapsto 3p_1 \wedge \frac{3}{2}p_2$, which is a (non-symmetric) p-merging function because for any p-variables P_1, P_2 and $\alpha \in (0, 1)$,

$$Q(F(P_1, P_2) \leq \alpha) \leq Q\left(P_1 \leq \tfrac{1}{3}\alpha\right) + Q\left(P_2 \leq \tfrac{2}{3}\alpha\right) \leq \tfrac{1}{3}\alpha + \tfrac{2}{3}\alpha = \alpha.$$

In the case $K = 3$, H_3^* is induced by the calibrator $3g$ on $(0, 1]$, where

$$g := 1_{[0,2/11]} + \tfrac{1}{2}1_{(2/11,4/11]} + \tfrac{1}{3}1_{(4/11,6/11]}.$$

Let the function F be given by the rejection set, for $\epsilon \in (0, 1)$,

$$R_\epsilon(F) = \epsilon\{\mathbf{p} \in [0, \infty)^3 : g_1(p_1) + g_2(p_2) + g_3(p_3) \geq 1\},$$

where $g_1 := g + \frac{1}{6}1_{(4/11,6/11]}$, $g_2 := g - \frac{1}{12}1_{(4/11,6/11]}$, and $g_3 := g_2$. By Theorem 5.1, F is a (non-symmetric) p-merging function. Direct calculation shows that F strictly dominates H_3^*.

Example 7.2 also shows that $H_K \wedge 1$ is not admissible for any $K \geq 2$, since it is either strictly dominated by H_K^* ($K \geq 4$) or by the functions in Example 7.2 ($K = 2, 3$).

7.2 Admissibility for the O-family

Next, we show that, except for the maximum merging function $G_{K,K}$, each member of the O-family is admissible if we trivially modify it by a zero-one adjustment, as in Proposition 2.2. Although $G_{K,K}$ fails to be admissible, it is admissible among symmetric p-merging functions after this modification.

Theorem 7.3. *The p-merging function*

$$\mathbf{p} \mapsto G_{k,K}(\mathbf{p} \wedge \mathbf{1}) \wedge 1_{\{\min(\mathbf{p})>0\}} = G_{k,K}(\mathbf{p}) \wedge 1_{\{\min(\mathbf{p})>0\}}$$

is admissible for $k = 1, \dots, K-1$, and it is admissible among symmetric p-merging functions for $k = K$.

Proof. As we see from Example 5.4, for each $k = 1, \dots, K$, $\mathbf{p} \mapsto G_{k,K}(\mathbf{p}) \wedge 1_{\{\min(\mathbf{p})>0\}}$ is induced by $f : x \mapsto \infty 1_{\{x=0\}} + (K/k)1_{\{x\in(0,k/K]\}}$.

First, fix $m = 1, \dots, K-1$. Using Proposition 6.1, suppose, for the purpose of contradiction, that there exist $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and calibrators $g_1, \dots, g_K$ satisfying (16). For each $k = 1, \dots, K$, denote $y_k := \lambda_k g_k(m/K)$. Since $1 = \int_0^1 g_k(x)\mathrm{d}x \geq \frac{m}{K} g_k(m/K)$, we have $y_k \leq \lambda_k K/m$, which implies $\sum_{k=1}^K y_k \leq K/m$.

Let Π_m be the set of all subsets of $\{1, 2, \dots, K\}$ of exactly m elements. There are $\binom{K}{m}$ elements (sets) in Π_m. For any $J \in \Pi_m$, take any $\beta > 1$ and let $\mathbf{p} = (p_1, \dots, p_K)$ be given by $p_k = \frac{m}{K} 1_{\{k\in J\}} + \beta 1_{\{k\notin J\}}$, $k = 1, \dots, K$. Since $\sum_{k=1}^K f(p_k) = K$, (16) implies $1 \leq \sum_{k=1}^K \lambda_k g_k(p_k) = \sum_{k\in J} y_k$. Therefore,

$$\binom{K}{m} \leq \sum_{J\in\Pi_m} \sum_{k\in J} y_k = \binom{K-1}{m-1} \sum_{k=1}^K y_k \leq \binom{K-1}{m-1} \frac{K}{m} = \binom{K}{m}.$$

This implies $\sum_{k\in J} y_k = 1$ for each $J \in \Pi_m$, and further $y_k = 1/m$ for each $k = 1, \dots, K$. Therefore, $\lambda_k \geq \int_0^{m/K} \lambda_k g_k(x)\mathrm{d}x \geq \frac{m}{K} y_k = 1/K$. Since $\sum_{k=1}^K \lambda_k = 1$, we have $g_k = f$ almost everywhere, which further implies $g_k \leq f$, and $\lambda_k = 1/K$, $k = 1, \dots, K$. Therefore, both sides of (16) coincide, which is a contradiction. Thus, $G_{m,K}(\mathbf{p}) \wedge 1_{\{\min(\mathbf{p})>0\}}$ is admissible for each $m = 1, \dots, K-1$.

To prove the statement for $m = K$, suppose that there exists a calibrator g satisfying (15). Since $f(x) = 1$ for $x \in (0, 1]$, we have $\sum_{k=1}^K f(x) = K$, which gives $K \leq Kg(x)$, and thus $g(x) \geq K/m = f(x)$. We have $\int_0^{m/K} g(x)\mathrm{d}x \geq \int_0^{m/K} f(x)\mathrm{d}x = 1$. As g is a calibrator, this means $g = f$ almost everywhere and further implies $g \leq f$. Therefore, both sides of (15) coincide, which is a contradiction. Thus, $G_{K,K}(\mathbf{p}) \wedge 1_{\{\min(\mathbf{p})>0\}}$ is admissible among symmetric p-merging functions. $\square$

8 The M-family

In this section, we study admissibility and the domination structure among the M-family of p-merging functions, which turn out to be drastically different from those of the O-family, as members in the M-family are generally not admissible, except for the cases of $F_{-\infty,K}$ and $F_{\infty,K}$ covered in Theorem 7.3. The key result of this section is Theorem 8.2, which gives another admissible p-merging function.

8.1 Coefficients in the M-family

To study functions $F_{r,K} = b_{r,K} M_{r,K} \wedge 1$ in the M-family, we first need to identify the constants $b_{r,K}$, which unfortunately do not always admit an analytical form. The values of $b_{r,K}$ are obtained in Vovk and Wang [2020a] for the cases $r \geq 1/(K-1)$ (Proposition 3), $r = 0$ (Proposition 4), and $r = -1$ (Proposition 6), where the proposition numbers refer to those in Vovk and Wang [2020a]. In addition, the values $b_{-\infty,K} = K$ and $b_{\infty,K} = 1$ are trivial to check. Below, we complement these results by providing formulas of $b_{r,K}$ for all $r \in \mathbb{R}$ via an analytical equation. We fix some notation which will be useful throughout this section. For a fixed K and $r \in (-\infty, 1/(K-1))$, let c_r be the unique number $c \in (0, 1/K)$ solving the equation

$$\begin{aligned} (K-1)(1-(K-1)c)^r + c^r &= K\frac{(1-(K-1)c)^{r+1} - c^{r+1}}{(r+1)(1-Kc)}, \quad \text{if } r \notin \{-1, 0\}; \\ \frac{1-Kc}{Kc(1-(K-1)c)} &= \log(1/c - (K-1)), \quad \text{if } r = -1; \\ K(1-Kc) &= \log(1/c - (K-1)), \quad \text{if } r = 0. \end{aligned}$$

The existence and uniqueness of the solution c to the above equation can be checked directly, and it is implied by Lemma 3.1 of Jakobsons et al. [2016] in a more general setting. Moreover, set $c_r := 0$ if $r \geq 1/(K-1)$, and write

$$d_r := 1 - (K-1)c_r, \quad r \in \mathbb{R}. \tag{25}$$

Notice that we always have $0 \leq c_r < 1/K < d_r \leq 1$.

The proofs of propositions in this section are put in Supplemental Article, Section A.3.

Proposition 8.1. *For $K \geq 2$ and $r \geq \frac{1}{K-1}$, we have $b_{r,K} = ((r+1) \wedge K)^{1/r}$. For $K \geq 3$ and $r \in (-\infty, \frac{1}{K-1})$, we have $b_{r,K} = 1/M_{r,K}(c_r, d_r, \ldots, d_r)$. For $r \in (-\infty, 1)$, we have $b_{r,2} = 2$.*

Via well-known inequalities on generalized mean functions [Hardy et al., 1952], it is straightforward to check, without using Proposition 8.1, that if $r < s$ and $rs > 0$, then

$$K^{1/s-1/r} b_{r,K} \leq b_{s,K} \leq b_{r,K}. \tag{26}$$

The relationship (26) conveniently gives, among other implications, the monotonicity of the mapping $r \mapsto b_{r,K}$ and its continuity except at 0. The continuity at 0 can be verified via Proposition 8.1.

8.2 Admissibility of the M-family and improvements

As illustrated by the numerical examples in Vovk and Wang [2020a] and Wilson [2020], the most useful cases of the M-family are those with $r \leq 0$. In particular, the *harmonic p-merging function* $F_{-1,K}$, which is a constant times the harmonic mean p-value of Wilson [2019] (truncated to 1), has a special role among the M-family, and it performs similarly to the Hommel function; see Chen et al. [2020]. On the other hand, the members $F_{r,K}$ for $r > 1$ are rarely useful in practice due to their heavy dependence on large realized p-values.

As we already mentioned, members of the M-family are generally not admissible, and we will construct dominating admissible functions. We briefly explain the main idea for the case $r<0$, as the other cases are similar. Using the equality $b^r_{r,K} = K(c_r^r + (K-1)d_r^r)^{-1}$ in Proposition 8.1, the rejection region of $F_{r,K}$ for $\epsilon \in (0,1)$ is given by

$$R_\epsilon(F_{r,K}) = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \frac{\sum_{k=1}^K p_k^r}{c_r^r+(K-1)d_r^r} \ge 1\right\} = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \sum_{k=1}^K \frac{p_k^r - d_r^r}{c_r^r - d_r^r} \ge 1\right\}$$

(see Example 5.6). The strictly convex function $x \mapsto K(x^r - d_r^r)/(c_r^r - d_r^r)$ is generally not a calibrator. Nevertheless, there is a simple modification which induces a p-merging function dominating $F_{r,K}$. Define the function

$$f_r : x \mapsto K\left(\frac{x^r - d_r^r}{c_r^r - d_r^r} \wedge 1\right)_+ .$$

We can check that each f_r is a calibrator. Let F_r^* be the p-merging function induced by f_r, that is,

$$R_\epsilon(F_r^*) = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \sum_{k=1}^K \left(\frac{p_k^r - d_r^r}{c_r^r - d_r^r}\right)_+ \ge 1\right\}, \qquad \epsilon\in(0,1). \tag{27}$$

It is clear that F_r^* dominates $F_{r,K}$. Moreover, the calibrator f_r satisfies (21) with $\eta = c_r$, which means that F_r^* is admissible by Theorem 6.2. In this way, an admissible p-merging function dominating $F_{r,K}$ is constructed.

In the next result, we give a rigorous statement of the above idea for all $r < K-1$, and show that the rejection regions of F_r^* have a very simple relationship to those of $F_{r,K}$. Remember that the minimum $\wedge$ of two vectors is understood component-wise.

Theorem 8.2. *For $K \ge 3$ and $r \in (-\infty, K-1)$, $F_{r,K}$ is strictly dominated by the p-merging function $F^*_{r,K}$ defined, for $\mathbf{p}\in(0,\infty)^K$ and $\epsilon\in(0,1)$, via*

$$F^*_{r,K}(\mathbf{p}) \le \epsilon \iff F_{r,K}(\mathbf{p}\wedge(\epsilon d_r \mathbf{1})) \le \epsilon \quad \text{or} \quad \min(\mathbf{p}) = 0, \tag{28}$$

where d_r is given in (25). Moreover, $F^*_{r,K}$ is admissible unless $r=1$.

The proof of the theorem will show that $F^*_{r,K} = F^*_r$.

Proof. We first address the case $r < 1/(K-1)$. Note that, for $r\in(0,1/(K-1))$,

$$R_\epsilon(F_{r,K}) = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \frac{\sum_{k=1}^K p_k^r}{c_r^r+(K-1)d_r^r} \le 1\right\} = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \sum_{k=1}^K \frac{p_k^r - d_r^r}{c_r^r - d_r^r} \ge 1\right\}$$

and

$$R_\epsilon(F_{0,K}) = \epsilon\left\{\mathbf{p}\in[0,\infty)^K : \sum_{k=1}^K \frac{\log p_k - \log d_0}{\log c_0 - \log d_0} \ge 1\right\},$$

which share a form very similar to the case $r<0$. Define the functions

$$f_r : x \mapsto K\left(\frac{x^r - d_r^r}{c_r^r - d_r^r}\wedge 1\right)_+ \quad \text{for } r\ne 0 \quad \text{and} \quad f_0 : x\mapsto K\left(\frac{\log x - \log d_0}{\log c_0 - \log d_0}\wedge 1\right)_+ .$$

We can check with Proposition 8.1 that

$$\int_{c_r}^{d_r} \frac{x^r - d_r^r}{c_r^r - d_r^r}\mathrm{d}x = \frac{1-Kc_r}{c_r^r - d_r^r}\left(\frac{c_r^r + (K-1)d_r^r}{K} - d_r^r\right) = \frac{1-Kc_r}{K},$$

which implies $\int_0^1 f_r(x)\,\mathrm{d}x = 1$, and similarly for $r = 0$. Hence, f_r is a calibrator, which further satisfies (21). As we explained above for the case $r < 0$, the p-merging function F_r^* induced by f_r strictly dominates $F_{r,K}$, and the admissibility of F_r^* follows from Theorem 6.2. Finally, comparing the conditions for $\mathbf{p} \in R_\epsilon(F_{r,K})$ and $\mathbf{p} \in R_\epsilon(F_r^*)$, i.e., if $r \neq 0$,

$$\sum_{k=1}^{K} \frac{(p_k/\epsilon)^r - d_r^r}{c_r^r - d_r^r} \geq 1 \quad \text{and} \quad \sum_{k=1}^{K} \left(\frac{(p_k/\epsilon)^r - d_r^r}{c_r^r - d_r^r}\right)_+ \geq 1,$$

the only difference is that any value p_k larger than $d_r\epsilon$ is treated as $d_r\epsilon$ by F_r^*. This implies $F_r^* = F_{r,K}^*$ for $F_{r,K}^*$ in (28). The case $r = 0$ is similar.

Next, we prove the statement for $r \in [1/(K-1), K-1)$. Using Proposition 8.1, $b_{r,K}^r = r+1$. Hence, the rejection region of $F_{r,K}$ for $\epsilon \in (0,1)$ is given by

$$R_\epsilon(F_{r,K}) = \epsilon \left\{\mathbf{p} \in [0,\infty)^K : \frac{r+1}{K}\sum_{k=1}^{K} p_k^r \leq 1\right\} = \epsilon \left\{\mathbf{p} \in [0,\infty)^K : \frac{1}{K}\sum_{k=1}^{K} g_r(p_k) \geq 1\right\},$$

where $g_r : x \mapsto (r+1)(1-x^r)/r$. Let $\tau = r/(r+1) \in [1/K, 1-1/K)$. Define a function $f_r : x \mapsto \tau^{-1}(1-x^r)_+$ for $x > 0$ and $f_r(0) = K$. It is clear that f_r is a calibrator by checking $\int_0^1 f_r(x)\,\mathrm{d}x = 1$. Since $f_r \geq g_r$, we know that the p-merging function F_r^* induced by f_r dominates $F_{r,K}$. The domination $F_r^* \leq F_{r,K}$ is strict since it is easy to find some $p_1, \dots, p_K \in (0,\infty)$ such that $\sum_{k=1}^K f_r(p_k) \geq K > \sum_{k=1}^K g_r(p_k)$. Moreover, for $r \neq 1$, f_r is either strictly convex or strictly concave on $(0,1)$ satisfying (21), and hence F_r^* is admissible by Theorem 6.2. The statement $F_r^* = F_{r,K}^*$ is analogous to the case $r < 1/(K-1)$. $\square$

As seen from the proof of Theorem 8.2, the calibrator f_r of $F_{r,K}^*$ is given by

$$\begin{aligned}
x &\mapsto K\left(\frac{x^r - d_r^r}{c_r^r - d_r^r} \wedge 1\right)_+ && \text{if } r < 1/(K-1) \text{ and } r \neq 0;\\
x &\mapsto K\left(\frac{\log x - \log d_0}{\log c_0 - \log d_0} \wedge 1\right)_+ && \text{if } r = 0;\\
x &\mapsto K 1_{\{x=0\}} + \frac{r+1}{r}(1-x^r)_+ && \text{if } r \in [1/(K-1), K-1).
\end{aligned}$$

Remark 8.3. Although in different disguises, the harmonic* calibrator $f := f_{-1}$ of $F_{-1,K}^*$ (which we refer to as the *harmonic* p-merging function*) and the grid harmonic calibrator (22) are remarkably similar: on the set $\{x > 0 : 0 < f(x) < K\}$, one of them takes the form $f(x) = a/x - b$, and the other one takes the form $f(x) = a/\lceil bx \rceil$ for some suitably chosen values of $a, b > 0$. In other words, the calibrator of $F_{-1,K}^*$ can be seen as a continuous version of that of H_K^*. Both calibrators are shown in Figure 1. In Section 10, we shall see that $F_{-1,K}^*$ and H_K^* perform similarly in our simulation experiments.

To approximate $F_{r,K}^*$, we can apply Algorithm 1 to the calibrator f_r. Remember that this algorithm computes an upper bound that approximates the true value with accuracy δ in time $O(-K\log\delta)$.

In the next proposition, we give an explicit formula for $F_{r,K}^*$ in Theorem 8.2. In what follows, $p_{(1)}, \dots, p_{(K)}$ are always the order statistics of components of $\mathbf{p}$, from the smallest to the largest, and $\mathbf{p}_m := (p_{(1)}, \dots, p_{(m)})$ is the vector of the m smallest components of $\mathbf{p}$.

Proposition 8.4. *For $K \geq 3$ and $\mathbf{p} \in [0,\infty)^K$, we have, if $r \in (-\infty, 1/(K-1))$,*

$$F^*_{r,K}(\mathbf{p}) = \left(\bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{M_{r,m}(c_r, d_r, \ldots, d_r)} \right) \wedge 1_{\{p_{(1)}>0\}}, \tag{29}$$

and, if $r \in [1/(K-1), K-1)$, with the convention $\cdot/0 = \infty$,

$$F^*_{r,K}(\mathbf{p}) = \left(\bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{(1 - \frac{rK}{(r+1)m})_+} \right) \wedge 1_{\{p_{(1)}>0\}}. \tag{30}$$

Proposition 8.4 allows us to compute $F^*_{r,K}(\mathbf{p})$ in time $O(K \log K)$. This is the time needed for sorting the elements of $\mathbf{p}$; the rest of the computations takes time $O(K)$ since $M_{r,m+1}(\mathbf{p}_{m+1})$ can be computed from $M_{r,m}(\mathbf{p}_m)$ in time $O(1)$, for any $m \in \{1, \ldots, K-1\}$.

The remaining functions $F_{r,K}$ for $r \geq K-1$ are all strictly dominated by the maximum merging function $F_{\infty,K}$, which will be discussed in Proposition 8.6 below. To summarize, except for the Bonferroni and the maximum p-merging functions, any other member of the M-family is not admissible among homogeneous symmetric p-merging functions. Nevertheless, for $r < K-1$, a simple modification in (28) leads to admissible p-merging functions based on the generalized mean, which has a stronger power than the original members of the M-family.

The (in-)admissibility of $F^*_{r,K}$ for $r = 1$ cannot be studied via Theorem 6.2 since the calibrator is neither strictly convex or strictly concave. A discussion of the technical challenges in this special case is provided in Remark A.6 in Supplemental Article, Section A.6.

8.3 Domination structure within the M-family

Next, we study the domination structure within the M-family of p-merging functions $F_{r,K}$, which are generally not admissible. It turns out that most members of the family are not comparable; however, for $K = 2$ or large r, there are some domination relationships among the members in the family. We note that $M_{s,K}$ and $M_{r,K}$ for $r \neq s$ are not proportional to each other, and hence the relations of domination among members of the M-family are all strict.

The following proposition gives a simple comparison for $aM_{r,K}$ and $bM_{s,K}$, where a, b are two positive constants, e.g., $a = b_{r,K}$ and $b = b_{s,K}$. Using this result, we can compare two p-merging functions that are not precise (but perhaps have simpler forms), such as the asymptotically precise p-merging functions in Vovk and Wang [2020a].

Proposition 8.5. *For $r < s$, $K \geq 2$, and $a, b \in (0,\infty)$, the following statements hold.*

(i) $aM_{r,K}$ dominates $bM_{s,K}$ if and only if $a \leq b$.

(ii) $bM_{s,K}$ dominates $aM_{r,K}$ if and only if $rs > 0$ and $aK^{-1/r} \geq bK^{-1/s}$.

Proposition 8.5 immediately implies that the asymptotically precise p-merging functions ($K \to \infty$) in Table 1 of Vovk and Wang [2020a] do not dominate each other.

Proposition 8.6. *Suppose $r \neq s$. If $K = 2$, $F_{r,K}$ is dominated by $F_{s,K}$ if and only if $1 \leq r < s$ or $s < r \leq 1$. If $K \geq 3$, $F_{r,K}$ is dominated by $F_{s,K}$ if and only if $K-1 \leq r < s$.*

As a consequence of Proposition 8.6, in addition to $F_{\infty,K}$, the members $F_{r,K}$ for $r < K-1$ are admissible within the M-family if $K \geq 3$, and the members for $r \in [K-1, \infty)$ are not. In the simple case $K = 2$, the only two admissible members in the M-family are $F_{-\infty,2}$ and $F_{\infty,2}$, and the arithmetic average $F_{1,2}$ is the worst, as it is strictly dominated by every other member of the M-family.

9 Magnitude of improvement

By focusing on some of the most important cases, in the following proposition (proved in Supplemental Article, Section A.4) we calculate four ratios measuring the improvement of the dominating p-merging functions over the standard ones in Theorems 7.1 and 8.2.

Proposition 9.1. *For $K \geq 3$, we have*

$$\inf_{\mathbf{p}\in(0,1]^K} \frac{F^*_{1,K}(\mathbf{p})}{F_{1,K}(\mathbf{p})} = \inf_{\mathbf{p}\in(0,1]^K} \frac{F^*_{0,K}(\mathbf{p})}{F_{0,K}(\mathbf{p})} = 0, \qquad \inf_{\mathbf{p}\in(0,1]^K} \frac{F^*_{-1,K}(\mathbf{p})}{F_{-1,K}(\mathbf{p})} = 1-(K-1)c_{-1},$$

$$\min_{\mathbf{p}\in(0,1]^K} \frac{H^*_K(\mathbf{p})}{H_K(\mathbf{p})} = \min\left\{t>0 : \sum_{k=1}^{K} \frac{1_{\{t\geq k/K\}}}{\lceil k/t \rceil} \geq 1\right\} =: \gamma_K.$$

Moreover, $c_{-1} \sim 1/(K\log K)$ and $\gamma_K \sim 1/\log K$ as $K \to \infty$.

In Proposition 9.1, there is a sharp contrast between the greatest improvement of $F^*_{-1,K}$ and that of H^*_K over their standard counterparts: asymptotically as $K \to \infty$, $F^*_{-1,K}$ can improve $F_{-1,K}$ only by a factor of $1 - 1/\log K \to 1$, while H^*_K can improve H_K by a significant factor of $1/\log K \to 0$. This observation is interesting especially seeing that H_K and $F_{-1,K}$ perform similarly in simulation scenarios (see, e.g., the simulation studies in Wilson [2020] and Chen et al. [2020]). Moreover, since $H_K = \ell_K S_K$ and $\gamma_K \sim 1/\log K \sim 1/\ell_K$, H^*_K performs similarly to the Simes function S_K for some input p-values $\mathbf{p}$, e.g., those with order statistics close to $(1,\ldots,K)$ times a constant (as can be seen from (35) in Supplemental Article), a situation that likely happens if the p-values are generated iid from a flat density around 0. This is remarkable as we see in Theorem 3.1 that all symmetric p-merging functions are dominated by S_K. See also the numerical illustrations in Section 10.

10 Simulation results

In this section, we compare the performance of p-merging functions via simulation. First, as a simple illustration, in Figure 2 we plot the cumulative distribution functions of $F(P_1,\ldots,P_K)$, where F is one of H_K, H^*_K, $F_{-1,K}$, $F^*_{-1,K}$, Bonferroni, or S_K. The Simes function S_K is used as a lower bound because it is the minimum of all symmetric p-merging functions (Theorem 3.1). The random variables $P_1,\ldots,P_K$ are generated following Vovk and Wang [2020b, Section 8], essentially using correlated z-tests. Overall we generate $K = 10^6$ observations x from the Gaussian models $N(\mu,1)$ in such a way that the correlation between any pair of observations is 0.9 (the correlation 0.9 is chosen for a better visibility of the comparison; other choices of the correlation give qualitatively similar results, except for the Bonferroni function, which performs better for small correlations when testing the global null; see Section C in Supplemental Article). An exception is the last observation,

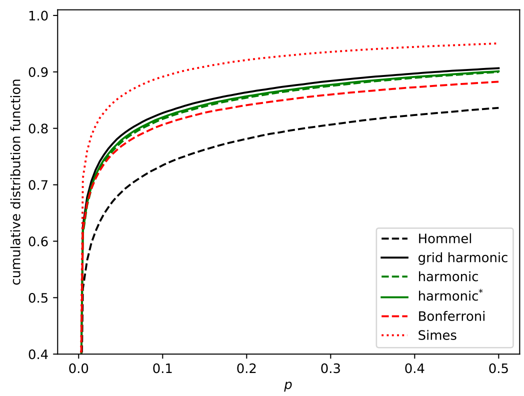

Figure 2: Cumulative distribution functions of $F(P_1, \ldots, P_K)$ for correlated z-tests (with correlation 0.9 between the vast majority of observations).

whose correlation with the other observations is -0.9. This violates the standard MTP_2 assumption [Sarkar, 1998], and so the application of the Simes test is not justified. (It is not justified anyway unless we *know* that MTP_2 holds; such knowledge is rare in practice.)

The null hypotheses are $N(0,1)$ and the alternatives are $N(-5,1)$. First we generate $K_1 = 10^3$ observations from the alternative distribution $N(-5,1)$ and then $K_0 := K - K_1$ observations from the null distribution $N(0,1)$. As the base p-values we take $P(x) := N(x)$, where N is the standard Gaussian distribution function. The empirical cumulative distribution function of $F(P_1, \ldots, P_K)$ is computed via an average of 10^5 independent simulations. A larger cumulative distribution function indicates greater power.

The Bonferroni and Hommel methods appear the worst and, of course, Simes is the best (but remember that it is not a valid method in our context). The other methods are roughly midway between these two. We can hardly distinguish between $F_{-1,K}$ and $F^*_{-1,K}$, but the grid harmonic method H^*_K performs somewhat better. In agreement with Proposition 9.1, the improvement of H^*_K over H_K is much more significant than the improvement of $F^*_{-1,K}$ over $F_{-1,K}$. Additional simulation results for discrete p-values are included in Section C of Supplemental Article.

Next let us see what our procedures give for multiple hypothesis testing. We will use a general procedure of Genovese and Wasserman [2004] and Goeman and Solari [2011], which we shall refer to as the GWGS procedure, and see how the new p-merging functions improve the performance over the classic ones.

Let $F = (F_k)_{k=1}^K$ be a family of symmetric p-merging functions, which will be chosen from the ones presented in Figure 2. Each F_k is a function of k p-variables, defined in the same way as its counterpart in Figure 2 but replacing K p-values by k p-values as its input. For any input p-values $\mathbf{p} = (p_1, \ldots, p_K)$ and any non-empty subset I of $\{1, \ldots, K\}$, we will write $F_{\mathbf{p}}(I)$ for the value of $F_{|I|}$ on a sequence consisting of $|I|$ elements p_i, $i \in I$ (in any order). With such an F and input p-values $\mathbf{p}$ we associate the array

$$\mathrm{DM}_{l,j} := \max_{I:|R\setminus I|<j} F_{\mathbf{p}}(I), \quad l \in \{1, \ldots, K\}, \quad j \in \{1, \ldots, l\}, \tag{31}$$

where $R \subseteq \{1, \ldots, K\}$ is a set of indices of l smallest p-values among $p_1, \ldots, p_K$ (such a set

Algorithm 2 Discovery matrix

Require: A family of merging functions F.
Require: An increasing sequence $\mathbf{p}$ of p-values $p_1 \le \cdots \le p_K$.

for $l = 1, \dots, K$ **do**
  for $j = 1, \dots, l$ **do**
    $S_{j,l} := \{j, \dots, l\}$
    $\mathrm{DM}'_{l,j} := F_{\mathbf{p}}(S_{j,l})$
    for $i = K, \dots, l+1$ **do**
      $p := F_{\mathbf{p}}(S_{j,l} \cup \{i, \dots, K\})$
      if $p > \mathrm{DM}'_{l,j}$ **then**
        $\mathrm{DM}'_{l,j} := p$

R may not be unique if there are ties among $p_1, \dots, p_K$, but $\mathrm{DM}_{l,j}$ does not depend on the choice of R). We regard DM as a $K \times K$ matrix whose elements above the main diagonal are undefined and call it the *(GWGS) discovery matrix*; this is our representation of the GWGS procedure. A small value of $\mathrm{DM}_{l,j}$ is evidence for the statement "there are at least j true discoveries among the l hypotheses (with the smallest p-values) that we choose to reject"; namely, $\mathrm{DM}_{l,j}$ is a valid p-value for testing the negation of this statement. These p-values are jointly valid in the sense that, for each confidence level $1-\alpha$, with probability at least $1-\alpha$, the maximum number j satisfying $\mathrm{DM}_{l,j} \le \alpha$ is a lower bound on the number of true discoveries among l smallest p-values for all l simultaneously.

See the recent paper Goeman et al. [2019] for an interesting justification of the GWGS procedure (it is the only admissible, in some sense, procedure with the true discovery guarantee). The goal of the GWGS procedure is somewhat similar to that of the partial conjunction test (see, e.g., Wang and Owen [2019]) looking for evidence that at least j out of l null hypotheses are false. The difference is that a GWGS matrix is jointly valid for all j and l (as described earlier), and the l null hypotheses are those with the smallest p-values.

Algorithm 2 computes the modification

$$\mathrm{DM}'_{l,j} := \max_{I:|R \setminus I| = j-1} F_{\mathbf{p}}(I), \quad l \in \{1, \dots, K\}, \quad j \in \{1, \dots, l\},$$

of the discovery matrix (31). It assumes, without loss of generality, that the input p-values are given in the increasing order. We will usually have $\mathrm{DM} = \mathrm{DM}'$, but unlike $\mathrm{DM}_{l,j}$, the function $\mathrm{DM}'_{l,j}$ does not need to be monotonically increasing in j. (The monotonicity may be violated when, e.g., F represents the Bonferroni p-merging functions.) But even in such unusual cases it is always true that $\mathrm{DM}_{l,j} = \max_{j' \le j} \mathrm{DM}'_{l,j'}$.

Figure 3 shows the upper left corners of size 120×120 of the discovery matrices produced by six of the p-merging functions considered in this paper for the p-variables $P_1, \dots, P_{1000}$ defined as before with the first 100 observations coming from the alternative distribution $N(-5,1)$ and the remaining 900 from the null distribution $N(0,1)$. It uses the standard significance levels 1% and 5% as thresholds; the values in the discovery matrices below 1% are shown in red, between 1% and 5% in yellow, and above 5% in green. As explained above, the number of red entries in the lth row of the discovery matrix is a lower bound on the number of true discoveries among l smallest p-values at the confidence level 99%, and the total number of red and yellow entries in the lth row is the analogous lower bound at the confidence level 95%.

The upper row of plots in Figure 3 shows the results for three standard methods, and

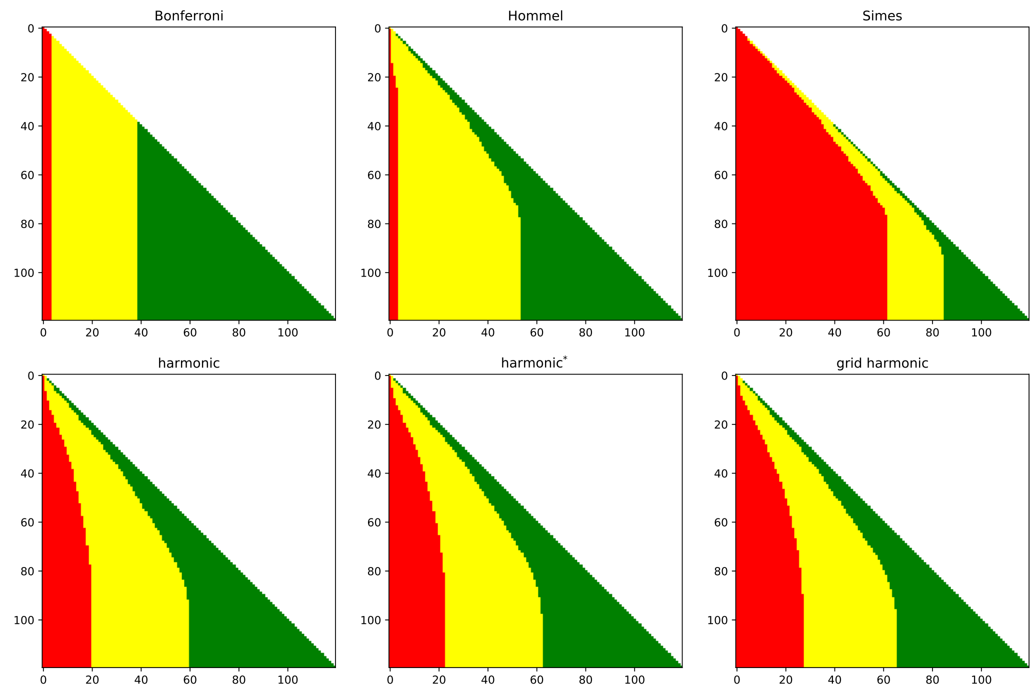

Figure 3: The GWGS discovery matrices for the simulation data using significance levels 1% and 5%. We give results for the p-merging functions $F_{-\infty,K}$ ("Bonferroni"), H_K ("Hommel"), S_K ("Simes"), $F_{-1,K}$ ("harmonic"), $F^*_{-1,K}$ ("harmonic*"), and H^*_K ("grid harmonic").

the lower row for three new methods. The two of the standard methods that are universally valid, Bonferroni and Hommel, perform worst. Harmonic averaging leads to better results. The results for $F^*_{-1,K}$ are better, but the difference is not substantial. The best results for a universally valid method are achieved by the grid harmonic merging function H^*_K. The results for the Simes merging function S_K are, of course, even better (in view of Theorem 3.1), but S_K is not valid in our setting.

Discovery matrices depend very much on the seed used for the pseudo-random number generator, especially for high correlations (such as 0.9 used in Figure 3). To make our results more reproducible, the discovery matrices in Figure 3 are in fact element-wise medians over 10 simulations. For other correlation coefficients, we obtain qualitatively similar results; see Section C in Supplemental Article.

11 Concluding remarks

In this paper, we establish a representation and some conditions for admissible p-merging functions via calibrators. Several new p-merging functions, most notably H^*_K and $F^*_{-1,K}$, are proposed and shown to be admissible. As seen from our main results and their proofs, admissibility of p-merging functions is a sophisticated object.

We mention a few open questions. First, our study is mainly confined to homogeneous p-merging functions. The homogeneity requirement in Theorem 5.1 is essential to our proof, and it is unclear whether or how one could relax it. On the other hand, most p-merging functions used in practice are homogeneous (an exception is the Cauchy combination test of Liu and Xie [2020], which is not valid for arbitrary dependence and hence does not fit into our setting). Second, it is unclear how the strict convexity in Theorem 6.2 can be relaxed; see discussions in Remark A.6. As a consequence, we suspect, but could not prove, the admissibility of $F^*_{1,K}$ for $K \geq 3$. This function is not admissible for $K = 2$; see Example 5.5. Third, we do not know whether H^*_K is always inadmissible for all prime numbers K (see Example 7.2 for the cases of $K = 2$ and $K = 3$). Fourth, an admissible p-merging function dominating a given p-merging function is typically not unique. We wonder whether there are other admissible p-merging functions which dominate H_K and $F_{-1,K}$, the two most important inadmissible p-merging functions, that have analytical formulas as well as superior statistical performance. Finally, it is important to develop more efficient ways of computing H^*_K; in our simulation studies we used a brute-force method based on Algorithm 1.

Author contributions

The author names are listed in the alphabetical order. The main mathematical results are due to Bin Wang and Ruodu Wang. Vladimir Vovk has contributed to the presentation and computational experiments.

Acknowledgments

The authors thank the Editor, an Associate Editor, and three anonymous referees of the journal version of this paper for very helpful comments. V. Vovk's research has been partially supported by Amazon, Astra Zeneca, and Stena Line. R. Wang is supported by the Natural Sciences and Engineering Research Council of Canada (RGPIN-2018-03823, RGPAS-2018-522590).

References

Yuyu Chen, Peng Liu, Ken Seng Tan, and Ruodu Wang. Trade-off between validity and efficiency of merging p-values under arbitrary dependence. Technical Report arXiv:2007.12366 [math.ST], arXiv.org e-Print archive, July 2020.

Bradley Efron. *Large-Scale Inference: Empirical Bayes Methods for Estimation, Testing, and Prediction*. Cambridge University Press, Cambridge, 2010.

Hans Föllmer and Alexander Schied. *Stochastic Finance: An Introduction in Discrete Time*. De Gruyter, Berlin, third edition, 2011.

Christopher Genovese and Larry Wasserman. A stochastic process approach to false discovery control. *Annals of Statistics*, 32(3):1035–1061, 2004.

Jelle J. Goeman and Aldo Solari. Multiple testing for exploratory research. *Statistical Science*, 26(4):584–597, 2011.

Jelle J. Goeman, Jesse Hemerik, and Aldo Solari. Only closed testing procedures are admissible for controlling false discovery proportions. Technical Report arXiv:1901.04885 [math.ST], arXiv.org e-Print archive, February 2019. To appear in the *Annals of Statistics*.

I. J. Good. Significance tests in parallel and in series. *Journal of the American Statistical Association*, 53:799–813, 1958.

Peter Grünwald, Rianne de Heide, and Wouter M. Koolen. Safe testing. Technical Report arXiv:1906.07801 [math.ST], arXiv.org e-Print archive, June 2020.

G. H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge University Press, Cambridge, second edition, 1952.

Gerhard Hommel. Tests of the overall hypothesis for arbitrary dependence structures. *Biometrical Journal*, 25:423–430, 1983.

Edwards Jakobsons, Xiaoying Han, and Ruodu Wang. General convex order on risk aggregation. *Scandinavian Actuarial Journal*, 2016:713–740, 2016.

Yaowu Liu and Jun Xie. Cauchy combination test: a powerful test with analytic p-value calculation under arbitrary dependency structures. *Journal of the American Statistical Association*, 115:393–402, 2020.

Bernhard Rüger. Das maximale Signifikanzniveau des Tests “Lehne H_0 ab, wenn k unter n gegebenen Tests zur Ablehnung führen”. *Metrika*, 25:171–178, 1978.

Ludger Rüschendorf. On the distributional transform, Sklar’s theorem, and the empirical copula process. *Journal of Statistical Planning and Inference*, 139:3921–3927, 2009.

Ludger Rüschendorf. *Mathematical Risk Analysis*. Springer, Heidelberg, 2013.

Sanat K. Sarkar. Some probability inequalities for ordered MTP_2 random variables: a proof of the Simes conjecture. *Annals of Statistics*, 26:494–504, 1998.

Glenn Shafer. The language of betting as a strategy for statistical and scientific communication. Technical Report arXiv:1903.06991 [math.ST], arXiv.org e-Print archive, March 2019. To appear in the *Journal of the Royal Statistical Society: Series A* (with discussion).

R. John Simes. An improved Bonferroni procedure for multiple tests of significance. *Biometrika*, 73:751–754, 1986.

Vladimir Vovk and Ruodu Wang. Combining p-values via averaging. *Biometrika*, 107: 791–808, 2020a.

Vladimir Vovk and Ruodu Wang. E-values: Calibration, combination, and applications. Technical Report arXiv:1912.06116 [math.ST], arXiv.org e-Print archive, September 2020b. To appear in the *Annals of Statistics*.

Vladimir Vovk and Ruodu Wang. A class of ie-merging functions. Technical Report arXiv:2007.06382 [math.ST], arXiv.org e-Print archive, July 2020c.

Bin Wang and Ruodu Wang. The complete mixability and convex minimization problems with monotone marginal densities. *Journal of Multivariate Analysis*, 102(10):1344–1360, 2011.

Bin Wang and Ruodu Wang. Joint mixability. *Mathematics of Operations Research*, 41(3): 808–826, 2016.

Jingshu Wang and Art B. Owen. Admissibility in partial conjunction testing. *Journal of the American Statistical Association*, 114:158–168, 2019.

Ruodu Wang, Liang Peng, and Jingping Yang. Bounds for the sum of dependent risks and worst Value-at-Risk with monotone marginal densities. *Finance and Stochastics*, 17: 395–417, 2013.

Daniel J. Wilson. The harmonic mean p-value for combining dependent tests. *Proceedings of the National Academy of Sciences*, 116:1195–1200, 2019.

Daniel J. Wilson. Generalized mean p-values for combining dependent tests: comparison of generalized central limit theorem and robust risk analysis. *Wellcome Open Research*, 5 (55):1–21, 2020.

Supplemental Article

A Technical details

A.1 Proofs of Propositions 2.1, 2.2, 2.3 and 2.4

Proof of Proposition 2.1. Suppose that F is an admissible p-merging function and there exists $b \in (0,1)$ such that

$$a := \sup_{\mathbf{P} \in \mathcal{P}_Q^K} Q(F(\mathbf{P}) \le b) < b.$$

Define the increasing function $h : [0,\infty) \to [0,\infty)$ by $h(x) := a1_{\{x \in [a,b]\}} + x1_{\{x \notin [a,b]\}}$. We can check, for $t \in [a,b]$,

$$\sup_{\mathbf{P} \in \mathcal{P}_Q^K} Q(h \circ F(\mathbf{P}) \le t) = \sup_{\mathbf{P} \in \mathcal{P}_Q^K} Q(F(\mathbf{P}) \le b) = a \le t,$$

and for $t \notin [a,b]$,

$$\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q(h\circ F(\mathbf{P})\le t) = \sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q(F(\mathbf{P})\le t)\le t.$$

Hence, $h\circ F$ is a p-merging function. The fact that $(h\circ F)\wedge 1$ strictly dominates $F\wedge 1$ contradicts the admissibility of F. Therefore, we obtain $\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q(F(\mathbf{P})\le t)\ge t$, $t\in(0,1)$. Together with the fact that F is a p-merging function, we have

$$\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q(F(\mathbf{P})\le t) = t, \quad t\in(0,1),$$

and thus F is precise. □

Proof of Proposition 2.2. Fix $\mathbf{P}=(P_1,\dots,P_K)\in\mathcal{P}_Q^K$ and $\alpha\in(0,1)$, and we will first show $Q(F'(\mathbf{P})\le\alpha)\le\alpha$. For every $\lambda\in(0,1)$, let A_λ be an event independent of $\mathbf{P}$ with $Q(A_\lambda)=\lambda$ and define the random vector $\mathbf{P}^\lambda=(P_1^\lambda,\dots,P_K^\lambda)$ via $\mathbf{P}^\lambda=\lambda\mathbf{P}$ if A_λ occurs, and $\mathbf{P}^\lambda=(1,\dots,1)$ if A_λ does not occur. For all $\lambda\in(0,1)$ and $k=1,\dots,K$, noting that $Q(P_k\le\alpha/\lambda)\le\alpha/\lambda$, we have

$$Q(P_k^\lambda\le\alpha)=\lambda Q(\lambda P_k\le\alpha)=\lambda Q(P_k\le\alpha/\lambda)\le\alpha.$$

Thus, $\mathbf{P}^\lambda\in\mathcal{P}_Q^K$ and by the fact that F is a p-merging function, we have $Q(F(\mathbf{P}^\lambda)\le\alpha)\le\alpha$. Note that

$$Q(F(\mathbf{P}^\lambda)\le\alpha)\ge Q(A_\lambda)Q(F(\mathbf{P}^\lambda)\le\alpha|A_\lambda)=\lambda Q(F(\lambda\mathbf{P})\le\alpha),$$

from which we obtain

$$Q(F(\lambda\mathbf{P})\le\alpha)\le\frac{\alpha}{\lambda}.$$

Since F is increasing, by (1), we have $F'(\mathbf{P})\ge F(\lambda\mathbf{P})$ for all $\lambda\in(0,1)$. Therefore,

$$Q(F'(\mathbf{P})\le\alpha)\le Q(F(\lambda\mathbf{P})\le\alpha)\le\frac{\alpha}{\lambda}.$$

Since $\lambda\in(0,1)$ is arbitrary, we have $Q(F'(\mathbf{P})\le\alpha)\le\alpha$, thus showing that F' is a p-merging function.

For the statement on $\tilde{F}$, it is clear that

$$Q\left(\mathbf{P}\in[0,1]^K\setminus(0,1]^K\right)=Q\left(\bigcup_{k=1}^K\{P_k=0\}\right)\le\sum_{k=1}^K Q(P_k=0)=0.$$

Therefore, the values of F on $[0,1]^K\setminus(0,1]^K$ do not affect its validity as a p-merging function.

To show the last statement, let F be an admissible p-merging function. Using the above results, we obtain that $F'\le F$ is a p-merging function. Admissibility of F forces $F=F'$, implying that F is lower semicontinuous. Similarly, $F=\widetilde{F}$, implying that F takes value 0 on $[0,1]^K\setminus(0,1]^K$. □

Proof of Proposition 2.3. Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of p-merging functions which converges to its point-wise limit F. For any $\mathbf{P}=(P_1,\dots,P_K)\in\mathcal{P}_Q^K$, we know that $F_n(\mathbf{P})\to F(\mathbf{P})$ in distribution. Using the Portmanteau theorem, we have for all $\alpha\in(0,1)$,

$$Q(F(\mathbf{P})<\alpha)\le\liminf_{n\to\infty}Q(F_n(\mathbf{P})<\alpha)\le\alpha.$$

It follows that for any $\epsilon > 0$ and $\alpha \in (0, 1)$,

$$Q(F(\mathbf{P}) \le \alpha) \le \alpha + \epsilon.$$

Since α and ϵ are arbitrary, we know that $F(\mathbf{P})$ is a p-variable, and F is a p-merging function. □

Proof of Proposition 2.4. Let R be the uniform probability measure on $[0, 1]^K$. Fix a p-merging function F. Set $F_0 := F$ and let

$$c_i := \sup_{G:G\le F_{i-1}} \int_0^1 R(G \le \epsilon)\,\mathrm{d}\epsilon, \tag{32}$$

where $i := 1$ and G ranges over all p-merging functions dominating F_{i-1}. Let F_i be a p-merging function satisfying

$$F_i \le F_{i-1} \quad \text{and} \quad \int_0^1 R(F_i \le \epsilon)\,\mathrm{d}\epsilon \ge c_i - 2^{-i}, \tag{33}$$

where $i := 1$. Continue setting (32) and choosing F_i to satisfy (33) for $i = 2, 3, \ldots$. Set $G := \lim_{i\to\infty} F_i$. By Proposition 2.3, G is a p-merging function. Clearly, G dominates F and

$$\int_0^1 R(G \le \epsilon)\,\mathrm{d}\epsilon = \int_0^1 R(H \le \epsilon)\,\mathrm{d}\epsilon$$

for any p-merging function H dominating G.

By Proposition 2.2, the zero-one adjusted version $\widetilde{G}$ of G is a p-merging function, and so is the lower semicontinuous version $\widetilde{G}'$ of $\widetilde{G}$. Clearly $\widetilde{G}' = 0$ on $[0, 1]^K \setminus (0, 1]^K$. Let us check that $\widetilde{G}'$ is admissible. Suppose that there exists a p-merging function H such that $H \le \widetilde{G}'$ and $H \ne \widetilde{G}'$ on $(0, 1]^K$. Fix such an H and a $\mathbf{p} \in (0, 1]^K$ satisfying $H(\mathbf{p}) < \widetilde{G}'(\mathbf{p})$. Since $\widetilde{G}'$ is lower semicontinuous and H is increasing, there exists $\lambda \in (0, 1)$ such that $H < \widetilde{G}'$ on the hypercube $[\lambda\mathbf{p}, \mathbf{p}] \subseteq [0, 1]^K$, which has a positive R-measure. This gives

$$\int_0^1 R(G \le \epsilon)\,\mathrm{d}\epsilon \le \int_0^1 R(\widetilde{G}' \le \epsilon)\,\mathrm{d}\epsilon < \int_0^1 R(H \le \epsilon)\,\mathrm{d}\epsilon,$$

a contradiction. □

A.2 Proof of Proposition 6.1 and a lemma used in the proof of Theorem 6.2

Proof of Proposition 6.1. We will only show the first statement, as the second one follows from essentially the same proof. It suffices to show that F is not admissible among symmetric p-merging functions if and only if (15) holds for some calibrator g. First, if there exists such g, then the p-merging function based on the calibrator g strictly dominates F. Second, if F is not admissible, using Proposition 2.4 and Remark 2.5, we know that there exists $G \le F$ that is admissible among symmetric p-merging functions. Note that G can be safely chosen as homogeneous. Using Theorem 5.2, G is induced by a calibrator g. Since G strictly dominates F, we know that (15) holds. □

Lemma A.1. *If the p-merging function induced by a calibrator f is admissible, then so is the p-merging function induced by g in (17) for any $\eta \in [0, 1/K]$.*

Proof of the lemma. The case $\eta = 0$ is trivial since $g = f$. If $\eta = 1/K$, then g induces the Bonferroni p-merging function, which is admissible as shown in Proposition 6.1 of Vovk and Wang [2020b]. Below we assume $\eta \in (0, 1/K)$. Let F and G be the p-merging functions induced by f and g, respectively, and let G' be a p-merging function dominating G. Suppose for the purpose of contradiction that there exists $\mathbf{p} \in [0,\infty)^K$ and $\alpha \in (0,1)$ such that $\alpha\mathbf{p} \in R_\alpha(G')$ and $\alpha\mathbf{p} \notin R_\alpha(G)$. Clearly, no component of $\mathbf{p}$ can be in $[0,\eta]$, and hence $\mathbf{p} \in (\eta,\infty)^K$. Let $\mathbf{p}' = (\mathbf{p} - \eta\mathbf{1})/(1 - K\eta)$. By the relationship between f and g, we know $\alpha\mathbf{p}' \notin R_\alpha(F)$. Let $A = R_\alpha(F) \cup \{\alpha\mathbf{p}'\}$. Take any vector $\mathbf{P}$ of p-variables, and let ν be the distribution of $\alpha((1 - K\eta)\mathbf{P} + \eta\mathbf{1})$. Further, let Π be the set of all permutations of the vector $(\alpha\eta, 1, \dots, 1)$ and μ be the discrete uniform distribution over Π. Clearly, $\Pi \subseteq R_\alpha(G) \subseteq R_\alpha(G')$. Let $\mathbf{P}'$ follow the distribution $(K\eta\alpha)\mu + K\eta(1-\alpha)\delta_{\mathbf{1}} + (1 - K\eta)\nu$. It is easy to verify that the components of $\mathbf{P}'$ are p-variables. Note that if $\alpha\mathbf{P} \in A$, then $\alpha((1 - K\eta)\mathbf{P} + \eta\mathbf{1}) \in (R_\alpha(G) \cup \{\alpha\mathbf{p}\}) \subseteq R_\alpha(G')$. We have

$$\begin{aligned}\alpha \geq Q(\mathbf{P}' \in R_\alpha(G')) &= K\eta\alpha + (1 - K\eta)Q(\alpha((1 - K\eta)\mathbf{P} + \eta\mathbf{1}) \in R_\alpha(G)) \\ &\geq K\eta\alpha + (1 - K\eta)Q(\mathbf{P} \in A).\end{aligned}$$

Hence, $Q(\mathbf{P} \in A) \leq \alpha$. Since $\mathbf{P}$ is arbitrary, this implies that the rejection region of F at level α can be enlarged to A, a contradiction of the admissibility of F. Therefore, the above $\mathbf{p}$ does not exist, and G is admissible. □

A.3 Proofs of Propositions 8.1, 8.4, 8.5 and 8.6

Proof of Proposition 8.1. The simple case $K = 2$ is discussed in Section B, and we assume $K > 2$. The cases $r \geq 1/(K-1)$, $r = -1$ and $r = 0$ are obtained in Propositions 3, 4, and 6 of Vovk and Wang [2020a], and are easily obtained from the case $r \notin \{-1, 0\}$ by letting $r \to -1$ or $r \to 0$, respectively. It remains to show the remaining cases. Let q_0 and q_1 be the essential infimum and the essential supremum of a random variable, respectively, and $\mathcal{U} \subset \mathcal{P}_Q$ be the set of U[0, 1] random variables. Note that

$$R_\epsilon(F_{r,K}) = \left\{\mathbf{p} \in [0,\infty)^K : M_{r,K}(\mathbf{p}) \leq \frac{\epsilon}{b_{r,K}}\right\} = \epsilon\left\{\mathbf{p} \in [0,\infty)^K : M_{r,K}(\mathbf{p}) \leq \frac{1}{b_{r,K}}\right\}.$$

From $R_\epsilon(F_{r,K})$, in order for $\sup_{\mathbf{P}\in\mathcal{P}_Q^K} Q(\mathbf{P} \in R_\epsilon(F_{r,K})) = \epsilon$, it is necessary and sufficient to choose

$$\frac{1}{b_{r,K}} = \inf_{\mathbf{P}\in\mathcal{P}_Q^K} q_1(M_{r,K}(\mathbf{P})),$$

Simple algebra gives, for $r < 0$,

$$b_{r,K}^{-1} = \left(\frac{1}{K}\sup\{q_0(U_1^r + \cdots + U_K^r) : U_1, \dots, U_K \in \mathcal{U}\}\right)^{1/r},$$

and for $r > 0$,

$$b_{r,K}^{-1} = \left(\frac{1}{K}\inf\{q_1(U_1^r + \cdots + U_K^r) : U_1, \dots, U_K \in \mathcal{U}\}\right)^{1/r}.$$

The rest of the proof is a direct consequence of Lemma A.2 below, which gives, for $r < 0$,

$$\sup\{q_0(U_1^r + \cdots + U_K^r) : U_1, \dots, U_K \in \mathcal{U}\} = (K-1)(1 - (K-1)c)^r + c^r,$$

and for $r \in (0, 1/(K-1))$,

$$\inf\{q_1(U_1^r + \cdots + U_K^r) : U_1, \dots, U_K \in \mathcal{U}\} = (K-1)(1-(K-1)c)^r + c^r,$$

where $c = c_r$. Therefore, $b_{r,K}^{-1} = M_{r,K}(c_r, d_r, \dots, d_r)$. □

Lemma A.2. *For any increasing convex function $f : [0,1) \to \mathbb{R}$ satisfying either $f(1) = \infty$ or $f(1) - f(0) > K \int_0^1 (f(u) - f(0))\,\mathrm{d}u$ where $f(1)$ is the limit of $f(x)$ as $x \uparrow 1$, we have*

$$\sup\{q_0(f(U_1) + \cdots + f(U_K)) : U_1, \dots, U_K \in \mathcal{U}\} = (K-1)f((K-1)c_F) + f(1-c_F),$$

where c_F is the unique solution $c \in (0, 1/K)$ to the following equation

$$(K-1)F^{-1}((K-1)c) + F^{-1}(1-c) = K\frac{\int_{(K-1)c}^{1-c} F^{-1}(y)\,\mathrm{d}y}{1-Kc}. \tag{34}$$

Proof of the lemma. The lemma is essentially Theorem 3.4 of Wang et al. [2013] applied to the probability level $\alpha = 0$, noting that any convex quantile function f can be approximated by distributions with a decreasing density. □

Proof of Proposition 8.4. We use the calibrators f_r mentioned after Theorem 8.2. We first consider $r < 0$. For $m = 1, \dots, K$ and $p_1, \dots, p_K > 0$, let $\mathbf{v}_m := (c_r, d_r, \dots, d_r) \in \mathbb{R}^m$, and we have

$$\sum_{k=1}^{m} \frac{p_{(k)}^r - d_r^r}{c_r^r - d_r^r} \geq 1 \iff M_{r,m}(\mathbf{p}_m) \leq M_{r,m}(c_r, d_r, \dots, d_r) = M_{r,m}(\mathbf{v}_m).$$

Hence,

$$\sum_{k=1}^{K} \left(\frac{p_{(k)}^r - d_r^r}{c_r^r - d_r^r}\right)_+ \geq 1 \iff \bigvee_{m=1}^{K} \left(\sum_{k=1}^{m} \frac{p_{(k)}^r - d_r^r}{c_r^r - d_r^r}\right) \geq 1 \iff \bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{M_{r,m}(\mathbf{v}_m)} \leq 1.$$

Using its calibrator f_r, for each $\epsilon \in (0,1)$, $F_r^*(\mathbf{p}) \leq \epsilon$ if and only if $\bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{M_{r,m}(\mathbf{v}_m)} \leq \epsilon$, and hence (29) holds. The case $r \in [0, 1/(K-1))$ is similar.

Next, consider the case $r \geq 1/(K-1)$. For $m = 1, \dots, K$ and $p_1, \dots, p_K > 0$, we have

$$\frac{1}{K}\sum_{k=1}^{m} \tau^{-1}(1 - p_{(k)}^r) \geq 1 \iff M_{r,m}(\mathbf{p}_m) \leq 1 - \frac{\tau K}{m}.$$

Hence,

$$\sum_{k=1}^{m} f_r(p_{(k)}^r) \geq K \iff \bigvee_{m=1}^{K} \left(\sum_{k=1}^{m} \tau^{-1}(1 - p_{(k)}^r)_+\right) \geq K \iff \bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{(1 - \tau K/m)_+} \leq 1.$$

Since F_r^* is induced by f_r, we have, for $\epsilon \in (0,1)$, $F_r^*(\mathbf{p}) \leq \epsilon$ if and only if either $\bigwedge_{m=1}^{K} \frac{M_{r,m}(\mathbf{p}_m)}{(1-\tau K/m)_+} \leq \epsilon$ or $p_{(1)} = 0$. Hence, (30) holds. □

Proof of Proposition 8.5.

(i) To show the "if" statement, it suffices to note again that $M_{r,K}(\mathbf{u}) \le M_{s,K}(\mathbf{u})$ for all $\mathbf{u} \in (0,\infty)^K$ and the above inequality is strict unless $\mathbf{u}$ has only one positive component [Hardy et al., 1952, Theorem 16]. Therefore, $aM_{r,K}$ (strictly) dominates $bM_{s,K}$. To show the "only if" statement, we note that $aM_{r,K}$ cannot dominate $bM_{s,K}$ if $a > b$ since $M_{r,K}$ and $M_{s,K}$ agree on vectors with equal components.

(ii) We first assume $0 < r < s$. To show the "if" statement, it suffices to note again that $K^{1/r}M_{r,K}(\mathbf{u}) \ge K^{1/s}M_{s,K}(\mathbf{u})$ for all $\mathbf{u} \in [0,\infty)^K$ and the above inequality is strict if $\mathbf{u}$ does not have equal components [Hardy et al., 1952, Theorem 19]. Therefore, $bM_{s,K}$ (strictly) dominates $aM_{r,K}$. To show the "only if" statement, we note that, if $aK^{-1/r} < bK^{-1/s}$,

$$F_{r,K}(1,0,\dots,0) = aK^{-1/r} < bK^{-1/s} = F_{s,K}(1,0,\dots,0),$$

and thus $bM_{s,K}$ cannot dominate $aM_{r,K}$ if $aK^{-1/r} < bK^{-1/s}$.

We next assume $r < s < 0$. To show the "if" statement, we first note that, using Hardy et al. [1952, Theorem 19], for all $\mathbf{u} \in (0,\infty]^K$,

$$K^{1/r}M_{r,K}(1/\mathbf{u}) = \frac{1}{K^{-1/r}M_{-r,K}(\mathbf{u})} \ge \frac{1}{K^{-1/s}M_{-s,K}(\mathbf{u})} = K^{1/s}M_{s,K}(1/\mathbf{u}),$$

and the above inequality is strict if at least one of the components of $\mathbf{u}$ is 0. Therefore, $bM_{s,K}$ strictly dominates $aM_{r,K}$ if $aK^{-1/r} \le bK^{-1/s}$. To show the "only if" statement, we note that, if $aK^{-1/r} < bK^{-1/s}$, we have

$$\lim_{\epsilon \downarrow 0} aM_{r,K}(1,1/\epsilon,\dots,1/\epsilon) = aK^{-1/r} < bK^{-1/s} = \lim_{\epsilon \downarrow 0} bM_{s,K}(1,1/\epsilon,\dots,1/\epsilon),$$

and thus $bM_{s,K}$ cannot dominate $aM_{r,K}$ if $aK^{-1/r} < bK^{-1/s}$.

Finally, we consider the case $rs \le 0$. If $r \le 0 < s$, then using simple properties of the averages, we have

$$M_{r,K}(0,1,\dots,1) = 0 < \left(\frac{K-1}{K}\right)^{1/s} = M_{s,K}(0,1,\dots,1).$$

If $r < s = 0$, we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} M_{r,K}(\epsilon^K,1,\dots,1) = \lim_{\epsilon \downarrow 0} \left(\frac{M_{r,K}(\epsilon^K,1,\dots,1)}{\epsilon}\right) = \lim_{\epsilon \downarrow 0} \left(\frac{\epsilon^{Kr}+K-1}{K\epsilon^r}\right)^{1/r} = 0,$$

whereas

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} M_{0,K}(\epsilon^K,1,\dots,1) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} M_{0,K}(\epsilon^K,1,\dots,1) = 1 > 0.$$

In either case, $bM_{0,K}$ cannot dominate $aM_{r,K}$.

Summarizing the above cases, $bM_{0,K}$ dominates $aM_{r,K}$ if and only if $aK^{-1/r} \ge bK^{-1/s}$ and $rs > 0$. □

Proof of Proposition 8.6. In this proof, we do not truncate our merging functions at 1. That is, we directly treat $F_{r,K} = b_{r,K}M_{r,K}$ without loss of generality, since the functions in the M-family are homogeneous. We say that two p-merging functions are not comparable if neither of them dominates the other one.

Using Table 1 of Vovk and Wang [2020a] (or Section B), the case $K = 2$ follows directly from Proposition 8.5 since $b_{r,2} = 2^{1/r}$ for all $r \in [-\infty, 1]$ and $b_{r,2} = 2$ for $r < 1$. We next study the case $K \geq 3$. Using Table 1 of Vovk and Wang [2020a], $b_{r,K} = K^{1/r}$ for $r \geq K-1$. Hence, by Proposition 8.5, $F_{r,K}$ is dominated by $F_{s,K}$ if $K-1 \leq r < s$. We next show that this is the only possible domination between $F_{r,K}$ and $F_{s,K}$.

First, for $r, s \in [(K-1)^{-1}, K-1]$, we have $b_{r,K} = (1+r)^{1/r}$. Clearly, $b_{r,K}$ is strictly decreasing in r, and hence Proposition 8.5 (i) implies that $F_{r,K}$ does not dominate $F_{s,K}$ for $r < s$. Moreover, we can calculate

$$\frac{b_{r,K}K^{-1/r}}{b_{s,K}K^{-1/s}} = \frac{\left(\frac{1+r}{K}\right)^{1/r}}{\left(\frac{1+s}{K}\right)^{1/s}} = \left(\frac{1+r}{1+s}\right)^{1/s}\left(\frac{1+r}{K}\right)^{1/r-1/s} < 1.$$

Therefore, $F_{s,K}$ does not dominate $F_{r,K}$ either. We thus know that $F_{s,K}$ and $F_{r,K}$ are not comparable in this case.

Next, we consider $s < r \leq (K-1)^{-1}$. To show that $F_{s,K}$ and $F_{r,K}$ are not comparable, by (26) and Proposition 8.5, it suffices to show $b_{r,K} \neq b_{s,K}$ and $b_{r,K}K^{-1/r} \neq b_{s,K}K^{-1/s}$. These can be shown by straightforward (although cumbersome) calculation from the explicit formulas in Proposition 8.1. An intuitive explanation is that the dependence structure of the vector $\mathbf{P}_r \in \mathcal{P}_Q^K$ which gives the precise probability $Q(F_{r,K}(\mathbf{P}_r) \leq \epsilon) = \epsilon$ is different across $r \in (-\infty, K-1]$ (see, e.g., Wang et al. [2013]). This leads to $Q(F_{s,K}(\mathbf{P}_r) \leq \epsilon) < \epsilon$ and $Q(F_{r,K}(\mathbf{P}_s) \leq \epsilon) < \epsilon$ for $s \neq r$, and hence the two p-merging functions cannot be compared.

The above arguments show that each $F_{r,K}$, $r < K-1$, is not comparable with $F_{s,K}$ for s in a neighbourhood of r. Finally, using Lemma A.3 below, we obtain that $F_{r,K}$ for $r \leq K-1$ is admissible within the M-family □

Lemma A.3. *If $F_{r,K}$ is not dominated by $F_{s,K}$ for any s in a neighbourhood of r, then $F_{r,K}$ is admissible within the M-family.*

Proof of Lemma A.3. Since $F_{r,K}$ is not dominated by any $F_{s,K}$ for s in a neighbourhood of r, we obtain from Proposition 8.5 (i) that $b_{r,K} > b_{s,K}$ for all $s > r$ using monotonicity of $b_{r,K}$ in (26). Similarly, $b_{r,K}K^{-1/r} < b_{s,K}K^{-1/s}$ for all $s < r$ with $rs > 0$. Using Proposition 8.5 (i) and (ii), we know that $F_{r,K}$ is not dominated by $F_{s,K}$ if $rs > 0$. Also, by Proposition 8.5 (ii), $F_{r,K}$ is not dominated by $F_{s,K}$ if $s < r$ and $rs \leq 0$. Therefore, $F_{r,K}$ is admissible within the M-family. □

A.4 Proof of Proposition 9.1

Proof of Proposition 9.1. Let $\boldsymbol{\epsilon} = (\epsilon, \dots, \epsilon, 1) \in \mathbb{R}^K$ and $\boldsymbol{\epsilon}' = (\epsilon, 1, \dots, 1) \in \mathbb{R}^K$ for some $\epsilon > 0$.

(i) By definition, $F_{1,K}(\boldsymbol{\epsilon}) \geq 2/K$ and $F^*_{1,K}(\boldsymbol{\epsilon}) \leq \frac{2K}{K-2}\epsilon \leq 6\epsilon$. Hence, $F^*_{1,K}(\boldsymbol{\epsilon})/F_{1,K}(\boldsymbol{\epsilon}) \to 0$ as $\epsilon \downarrow 0$.

(ii) By definition, $F_{0,K}(\boldsymbol{\epsilon}') = \epsilon^{1/K}c$ for some constant $c > 0$ and $F^*_{0,K}(\boldsymbol{\epsilon}') \leq \epsilon c'$ for some constant $c' > 0$. Hence, $F^*_{0,K}(\boldsymbol{\epsilon}')/F_{0,K}(\boldsymbol{\epsilon}') \to 0$ as $\epsilon \downarrow 0$.

(iii) Write $c := c_{-1}$. For any $\mathbf{p} \in (0, \infty)^K$, we have

$$\frac{F^*_{-1,K}(\mathbf{p})}{F_{-1,K}(\mathbf{p})} = \bigwedge_{m=1}^{K} \frac{M_{-1,m}(\mathbf{p}_m)/M_{-1,m}(\mathbf{v}_m(c))}{M_{-1,K}(\mathbf{p})/M_{-1,K}(\mathbf{v}_K(c))}$$

$$= \bigwedge_{m=1}^{K} \left(\frac{c^{-1} + (m-1)(1-(K-1)c)^{-1}}{c^{-1} + (K-1)(1-(K-1)c)^{-1}} \times \frac{\sum_{k=1}^{K} p_{(k)}^{-1}}{\sum_{k=1}^{m} p_{(k)}^{-1}} \right)$$
$$\geq \bigwedge_{m=1}^{K} \frac{1-(K-1)c + (m-1)c}{1-(K-1)c+(K-1)c} = 1-(K-1)c,$$

where $\mathbf{v}_m(c) := (c, d, \dots, d)$ with $m-1$ entries of $d := 1-(K-1)c$. Taking $\mathbf{p} = \boldsymbol{\epsilon}'$ and letting $\epsilon \downarrow 0$ justifies the infimum value.

(iv) Take any $\mathbf{p}$ and let $\alpha = \bigwedge_{k=1}^{K} p_{(k)}/k$. Without loss of generality, we assume $\alpha K \ell_K \leq 1$ and hence $H_K^*(\mathbf{p}) \leq H_K(\mathbf{p}) \leq 1$. Since H_K^* is homogeneous, symmetric and increasing, we have

$$H_K^*(\mathbf{p}) \geq H_K^*(\alpha, 2\alpha, \dots, K\alpha) = \alpha K \ell_K \gamma_K = \gamma_K H_K(\mathbf{p}). \tag{35}$$

The minimum ratio $H_K^*(\mathbf{p})/H_K(\mathbf{p}) = \gamma_K$ is attained by $\mathbf{p} = (\alpha, 2\alpha, \dots, K\alpha)$ for $\alpha \in (0, 1/K\ell_K]$.

(v) We continue to write $c = c_{-1}$. Proposition 6 of Vovk and Wang [2020a] gives that $b_{-1,K} \sim \log K$, and with Proposition 8.1 we get $c(1-(K-1)c) \sim 1/(K \log K)$. Since $c \in (0, 1/K)$, the above implies $Kc \to 0$ as $K \to \infty$, and this further implies $c \sim 1/(K \log K)$. Next, we look at the quantity

$$y_K := \frac{1}{\gamma_K} = \max\left\{ y \geq 1 : \sum_{k=1}^{K} \frac{1_{\{y \leq K/k\}}}{\lceil ky \rceil} \geq 1 \right\}.$$

Note that $y' := \lfloor y_K \rfloor + 1$ satisfies $\sum_{k=1}^{K} \frac{1_{\{y' \leq K/k\}}}{\lceil ky' \rceil} < 1$, and we get

$$1 > \sum_{i=1}^{K} \frac{1_{\{y' \leq K/k\}}}{ky'} = \frac{1}{y'} \ell_{\lfloor K/y' \rfloor} \geq \frac{\log K - \log y'}{y'},$$

where the last inequality is due to $\ell_k \geq \log(k+1)$ for all $k \in \mathbb{N}$. Hence, $y' + \log y' > \log K$, which implies $y' > \log K - \log\log K$ and thus $y_K \geq \lfloor \log K - \log\log K \rfloor$. On the other hand, Theorem 3.1 implies that $y_K \leq H_K/S_K = \ell_K \leq \log K + 1$. Therefore, $y_K \sim \log K$ as $K \to \infty$. □

A.5 Naive procedure for merging p-values

As we saw in Section 4, p-to-e merging is easy. We can restate it formally as follows.

Corollary A.4. *The class of admissible p-to-e merging functions coincides with the class of functions (6), $f_1, \dots, f_K$ ranging over the admissible calibrators and $(\lambda_1, \dots, \lambda_K)$ over Δ_K.*

Proof. Combine Proposition 4.1 with a slightly generalized version (with the same proof) of Vovk and Wang [2020b, Proposition G.2]. □

A dual notion to p-to-e calibrators is that of e-to-p calibrators; the latter are functions that transform e-variables into p-variables. It turns out that the only admissible e-to-p calibrator is the reciprocal function $p \mapsto 1/p$ [Vovk and Wang, 2020b, Proposition 2.2]. The ease of merging e-values suggests merging p-values using a detour via e-values: (i) calibrate p-values $p_1, \dots, p_K$ via calibrators $f_1, \dots, f_K$ getting e-values $f_k(p_k)$; (ii) merge the e-values via weighted arithmetic average, getting $\sum_k \lambda_k f_k(p_k)$; (iii) calibrate the resulting e-value back to the p-value

$$F(p_1, \dots, p_K) := \frac{1}{\sum_k \lambda_k f_k(p_k)}. \tag{36}$$

This detour via e-values is in fact a poor procedure; e.g., the p-merging function (36) is not admissible. Let us check this.

To check that (36) is not admissible, suppose (temporarily allowing $K = 1$), without loss of generality, that all λ_k are positive and that all f_k are admissible and so upper semicontinuous. Arguing indirectly, suppose (36) is admissible and $c > 1$. We then have

$$\sup_P P\left(\left\{(p_1, \dots, p_K) \in [0,1]^K : \sum_k \lambda_k f_k(p_k) \geq c\right\}\right) = \frac{1}{c}, \tag{37}$$

P ranging over the probability measures on $[0,1]^\infty$ with the uniform marginals. Since this is true for any c, at least one of the f_k is unbounded on $(0,1]$. Now let us fix a $c > 1$. Since the set of probability measures P is compact in the topology of weak convergence and the set in (37) is closed, the supremum in (37) is attained, and so $\sum_k \lambda_k f_k(p_k) = c$ P-a.s.; this contradicts one of the f_k being unbounded on $(0,1]$.

As we can see, the naive procedure does not produce useful p-merging functions, but it turns out that it can be repaired. In the following somewhat informal argument we will ignore issues of measurability. To recover any p-merging function, it suffices to perform the detour via e-values for each rejection region (8) separately. Namely, for any $\epsilon \in (0,1)$: (i) calibrate p-values $p_1, \dots, p_K$ via calibrators $f_{1,\epsilon}, \dots, f_{K,\epsilon}$ getting e-values $e_{k,\epsilon} = f_{k,\epsilon}(p_k)$. (ii) Merge the e-values via weighted arithmetic average, getting $e_\epsilon = \sum_k \lambda_{k,\epsilon} e_{k,\epsilon}$. (iii) Include $(p_1, \dots, p_K)$ in R_ϵ if $1/e_\epsilon \leq \epsilon$. If $f_{k,\epsilon}$ are chosen in such a way that R_ϵ is increasing in ϵ, this will be a p-merging family (in the sense of satisfying $Q(\mathbf{P} \in R_\epsilon) \leq \epsilon$ for all $\epsilon \in (0,1)$ and $\mathbf{P} \in \mathcal{P}_Q^K$). And vice versa, by the duality theorem in the form of Proposition 4.1, for any p-merging function F and any $\epsilon \in (0,1)$, the rejection region $R_\epsilon(F)$ will be rejected in the sense

$$(p_1, \dots, p_K) \in R_\epsilon(F) \Longrightarrow \frac{1}{\sum_k \lambda_{k,\epsilon} f_{k,\epsilon}(p_k)} \leq \epsilon$$

for suitably chosen $f_{k,\epsilon}$ and $\lambda_{k,\epsilon}$.

The conclusion of Proposition 4.1, as applied to F that is constant in a region R and zero outside R, can be strengthened if we assume that R is a rejection region of an admissible p-merging function. The proof of Theorem 5.1 also proves the following proposition.

Proposition A.5. *For any admissible p-merging function F and $\epsilon \in (0,1)$, there exist $(\lambda_1, \dots, \lambda_K) \in \Delta_K$ and admissible calibrators $f_1, \dots, f_K$ such that*

$$F(\mathbf{p}) \leq \epsilon \iff \sum_{k=1}^K \lambda_k f_k(p_k) \geq \frac{1}{\epsilon}.$$

If F is symmetric, then there exists an admissible calibrator f such that

$$F(\mathbf{p}) \leq \epsilon \iff \frac{1}{K} \sum_{k=1}^K f(p_k) \geq \frac{1}{\epsilon}.$$

Let us specialize the modified naive procedure to homogeneous p-merging functions. According to Theorem 5.1, in the homogeneous case we can use calibrators $f_{k,\epsilon}(x) := f_k(x/\epsilon)/\epsilon$. The procedure becomes almost as simple as the naive procedure; both depend on a sequence $f_1, \dots, f_K$ of calibrators as parameter. If we are interested in homogeneous and symmetric p-merging functions, the detour via e-values can use calibrators $f_1 = \dots = f_K$ and the arithmetic mean as e-merging function (Theorem 5.2).

A.6 An additional technical remark on Theorems 6.2 and 8.2

Remark A.6. We discuss technical challenges arising in trying to relax the strict convexity (or strict concavity) imposed in Theorem 6.2 and to prove the admissibility of $F^*_{1,K}$ in Theorem 8.2 for $K \geq 3$. Recall in the proof of Theorem 6.2 that the density h is obtained from a distribution with quantile function f, and h is decreasing if f is convex. A crucial step in this proof is to verify that the distributions with densities $h_1, \dots, h_K$ are jointly mixable, which ensures that in (19), if A happens, the vector $(P_1, \dots, P_K)/\alpha = (f^{-1}(X_1), \dots, f^{-1}(X_K))$ satisfies $\sum_{k=1}^K f(P_k) \geq K$, so that $(P_1, \dots, P_K) \in R_\alpha(F)$. The densities $h_1, \dots, h_K$ are obtained from the density h by removing a tiny piece $m^* v_k/m_k$ for each k; see (18). Since $m^* v_k/m_k$ is tiny, the resulting density is still decreasing (or increasing) if h is strictly decreasing (or strictly increasing), and hence joint mixability can be obtained from Theorem 3.2 of Wang and Wang [2016]. In case the convex function f is linear on some interval (which is the case for $F^*_{1,K}$), h is constant on this interval. After removing a tiny piece on this interval from h, the resulting density is no longer monotone, and no result for joint mixability is available in this case. Proving joint mixability is known to be a very difficult task, although we suspect that it holds true for the above special case (if a proof is available, it likely will require a new paper). Unfortunately, it seems to us that one could not avoid this task for a generalization of Theorem 6.2, since showing $\sum_{k=1}^K f(P_k) \geq K$ for h with some pieces removed is essential for constructing any counter-example, at least to the best of our imagination.

B The case $K = 2$

In the simple case $K = 2$, where the task is to merge two p-values, the class of admissible p-merging functions admits an explicit description.

For $E \subseteq [0,1]^K$, let us set

$$\mathbb{P}(E) := \sup_{\mathbf{P} \in \mathcal{P}_Q^K} Q(\mathbf{P} \in E)$$

and call $\mathbb{P}(E)$ the *upper p-probability* of E. In the case $K = 2$ upper p-probability admits a simple characterization.

Lemma B.1. *The upper p-probability of any nonempty Borel lower set $E \subseteq [0,1]^2$ is*

$$\mathbb{P}(E) = 1 \wedge \inf \left\{ u_1 + u_2 : (u_1, u_2) \in [0,1]^2 \setminus E \right\}. \tag{38}$$

Proof. Let E be a nonempty lower Borel set in $[0,1]^2$; suppose $\mathbb{P}(E)$ is strictly less than the right-hand side of (38). Let t be any number strictly between $\mathbb{P}(E)$ and the right-hand side of (38). If $\mathbf{P}$ is concentrated on

$$[(t,0),(0,t)] \cup [(t,t),(1,1)], \tag{39}$$

and each of its components is uniformly distributed on $[0,1]$, $\mathbf{P} \in E$ with probability at least t since E contains $[(t,0),(0,t)]$. Therefore, $\mathbb{P}(E) \geq t$. This contradiction proves the inequality $\geq$ in (38).

As for the opposite inequality, we will check

$$\mathbb{P}(E) \leq \inf \left\{ u_1 + \cdots + u_K : (u_1, \ldots, u_K) \in [0,1]^K \setminus E \right\}$$

for an arbitrary $K \geq 2$. Let us assume that E does not contain the set of all $(u_1, \ldots, u_K)$ with $u_1 + \cdots + u_K = 1$ (the case when it does is trivial). Choose $\epsilon > 0$ and $(p_1, \ldots, p_K) \in [0,1]^K \setminus E$ such that $t := p_1 + \cdots + p_K \in [\epsilon, 1]$ and E contains all $(u_1, \ldots, u_K) \in [0,1]^K$ satisfying $u_1 + \cdots + u_K = t - \epsilon$. Since E is a lower set, we have

$$E \subseteq \bigcup_{k=1}^{K} \left\{ (u_1, \ldots, u_K) \in [0,1]^K : u_k \leq p_k \right\},$$

and the subadditivity of $\mathbb{P}$ further implies

$$\begin{aligned}
\mathbb{P}(E) &\leq \sum_{k=1}^{K} \mathbb{P}\left(\left\{ (u_1, \ldots, u_K) \in [0,1]^K : u_k \leq p_k \right\} \right) \\
&= \sum_{k=1}^{K} p_k = t \leq \inf \left\{ u_1 + \cdots + u_K : (u_1, \ldots, u_K) \in [0,1]^K \setminus E \right\} + \epsilon.
\end{aligned}$$

It remains to notice that ϵ can be chosen arbitrarily small. □

There is a natural bijection between the admissible p-merging functions for $K = 2$ and increasing right-continuous functions $f : [0,1) \to [0,1]$. The *epigraph boundary* of such f is the set of points $(u_1, u_2) \in [0,1]^2$ such that $f(u_1-) \leq u_2 \leq f(u_1)$, where $f(0-)$ is understood to be 0 and $f(1)$ is understood to be 1. A *diagonal curve* is the epigraph boundary of some increasing function. The admissible p-merging function corresponding to a diagonal curve $A \subseteq [0,1]^2$ is defined by $F(p_1, p_2) := u_1 + u_2$, where $(u_1, u_2) \in A$ is the largest point in A that is less than or equal to (p_1, p_2) in the component-wise order (A is linearly ordered by this partial order).

In particular, the only symmetric admissible p-merging function for $K = 2$ is Bonferroni. It corresponds to the identity function $f : u \mapsto u$.

C Additional simulation results

In this section we report some additional simulation results. Figure 4 is an analogue of Figure 2 with correlations 0.5 and 0 in place of 0.9, and Figures 6 and 7 are analogues of Figure 3 with correlations 0.5 and 0, respectively. One interesting phenomenon is that the performance of the Bonferroni method improves as we approach independence. The performance of the Bonferroni method also typically improves when there are fewer observations from the alternative hypothesis: see Figure 5, where we have 0.1% of observations from the alternative distribution in the left panel (which coincides with Figure 2) and 1% of observations from the alternative distribution in the right panel.

For other values of parameters (correlation, signal strength, signal sparsity, number of p-values) that we tried, the relative performance of the four methods that are our main

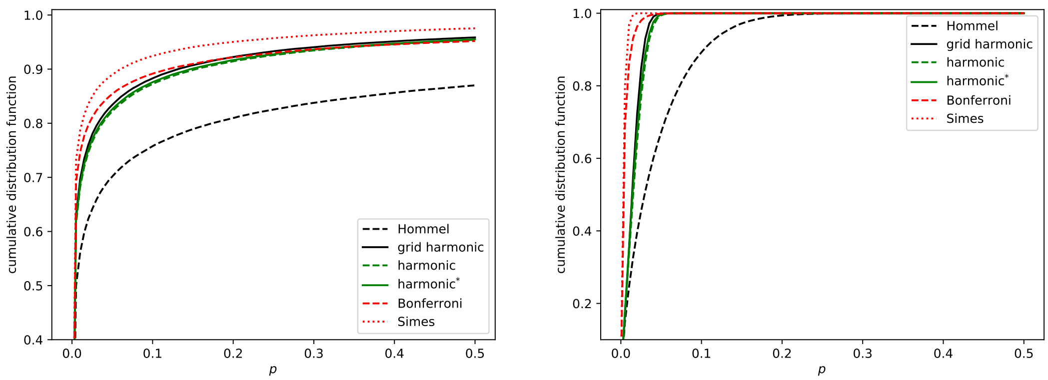

Figure 4: An analogue of Figure 2 for 10% of observations from the alternative distribution with correlation 0.5 (left panel) and 0 (right panel) in place of 0.9.

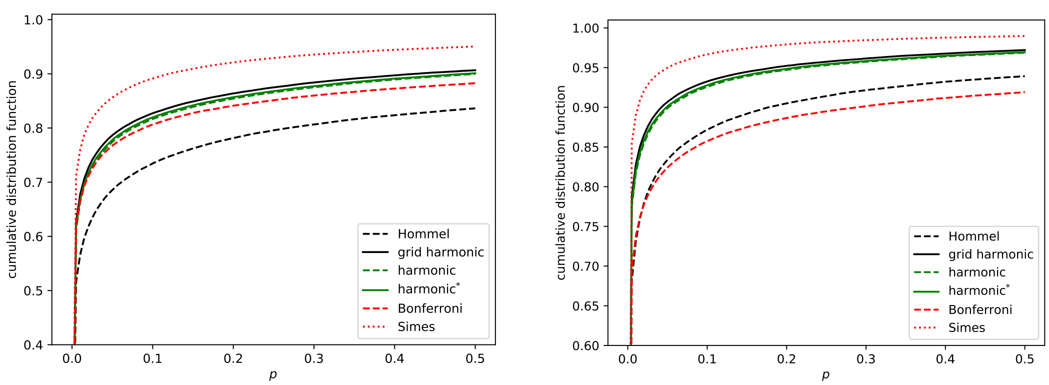

Figure 5: Figure 2 (left panel), where $K_1 = 10^3$ and the correlation is 0.9 for the bulk of the observations, and its counterpart with $K_1 := 10^4$ (right panel).

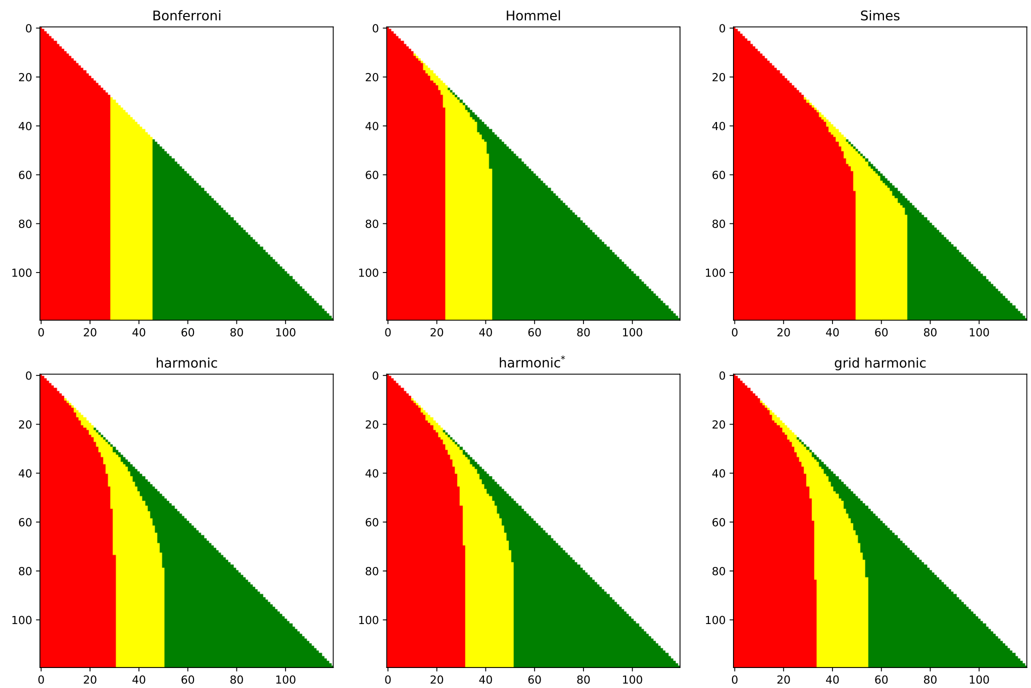

Figure 6: An analogue of Figure 3 with correlation 0.5: GWGS discovery matrices for the simulation data.

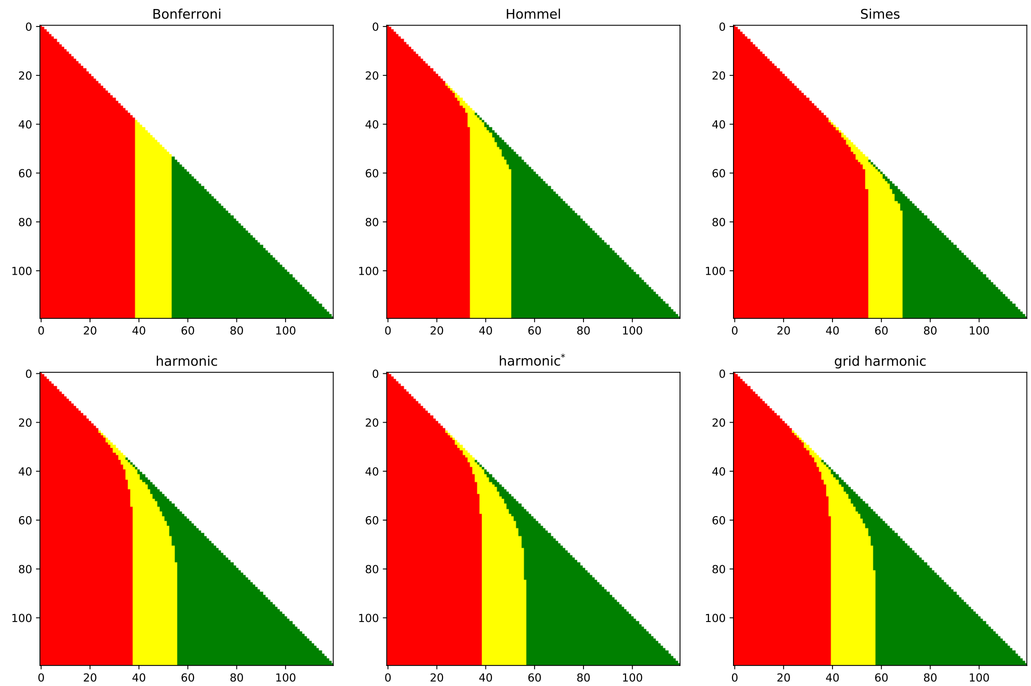

Figure 7: An analogue of Figure 3 with correlation 0.

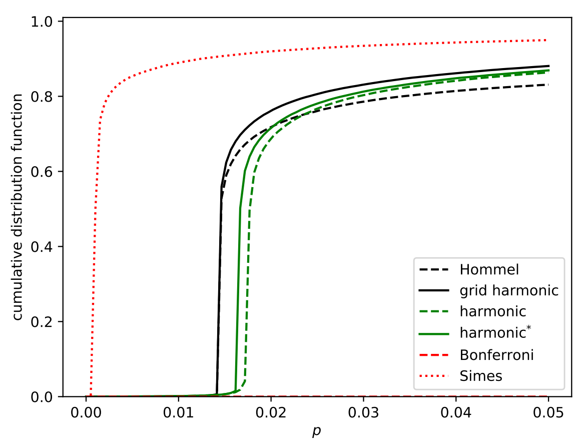

Figure 8: An analogue of Figure 2 for discrete p-values, as described in text.

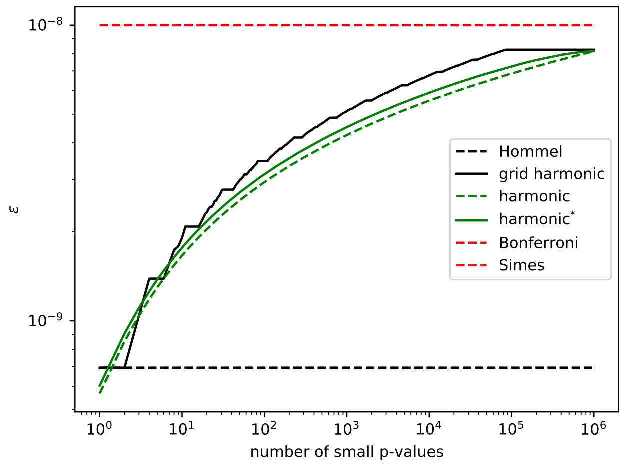

Figure 9: The smallest p-value leading to the combined p-value of 1%, as described in text, for various merging methods.

object of study (Hommel, grid harmonic, harmonic, and harmonic*) is qualitatively similar to the figures presented here and in Section 10.

Figures 8 and 9 illustrate some specifics of merging discrete p-values. Figure 8 is produced in the same way as Figure 2, except that each input p-value p is replaced by $\lceil Dp \rceil / D$, where we take $D := 10^4$. Now the Bonferroni function performs poorly; the corresponding curve is barely visible and coincides with the horizontal axis (our definition (5) gives a combined p-value of 1). We show only the most interesting part of the plot, for $p \in [0, 0.05]$. For small values of p Hommel's p-merging function is now better than the harmonic and even harmonic*.

In Figure 9 we again consider a set of $K = 10^6$ p-values generated by a test (e.g., a rank test) that produces p-values divisible by $\epsilon > 0$. A number $K_1 \in \{1, \ldots, K\}$ of these p-values are "small" (intuitively, correspond to a global null hypothesis being violated), and the remaining $K_0 := K - K_1$ p-values are 1. The small p-values are $\epsilon, 2\epsilon, \ldots, K_1\epsilon$. The question that we ask in this toy scenario is: how small should ϵ be in order for the combined p-value to be highly statistically significant?

Figure 9 gives the borderline values of ϵ (leading to the combined p-value of 1%) as function of K_1 for six merging methods. In this situation the Simes and Bonferroni methods produce the same borderline ϵ of 10^{-8} for all K_1. These are the best results (in this context the higher the better), while Hommel's method produces the worst result, 6.94×10^{-10}. The graphs for the remaining merging methods are instructive in that, whereas the grid harmonic method usually produces better results than harmonic and harmonic*, the shape of its graph is much less regular. While the discreteness of the grid harmonic calibrator (22) is not noticeable in our previous figures, in this combination with discrete p-values it becomes obvious. In the middle of the plot, $K_1 := 10^3$, the borderline values of ϵ are 5.12×10^{-9} for the grid harmonic method, 4.25×10^{-9} for harmonic, and 4.52×10^{-9} for harmonic*.